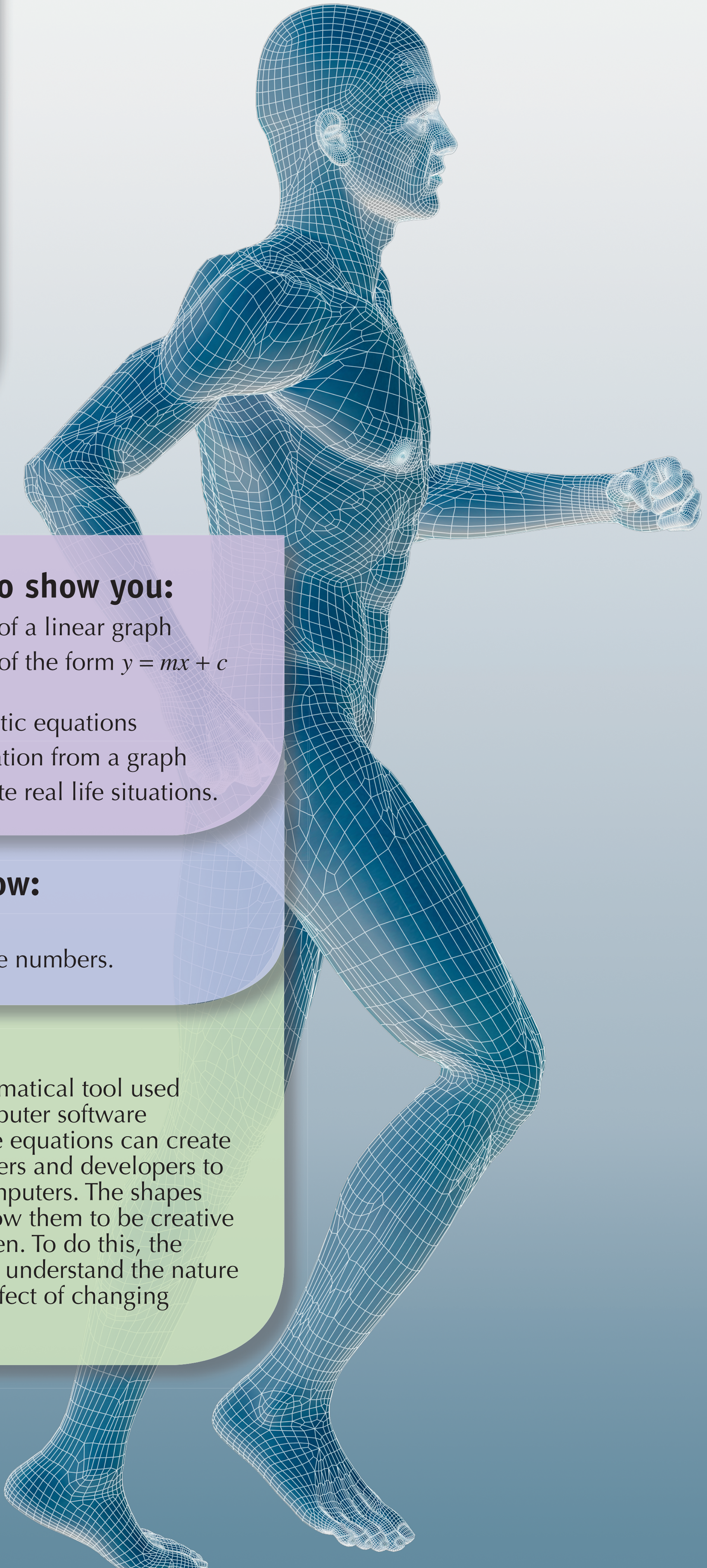
7

Graphs



This chapter is going to show you:

- how to work out the gradient of a linear graph
- how to work out an equation of the form $y = mx + c$ from its graph
- how to draw graphs of quadratic equations
- how to solve a quadratic equation from a graph
- how to draw graphs to illustrate real life situations.

You should already know:

- how to draw a linear graph
- how to calculate with negative numbers.

About this chapter

Equations are a powerful mathematical tool used in design, engineering and computer software development. The fact that these equations can create graphs allows designers, engineers and developers to model real life scenarios on computers. The shapes made by different equations allow them to be creative and to show movement on screen. To do this, the designers and engineers have to understand the nature of different equations and the effect of changing variables in them.

7.1 Graphs from linear equations

Learning objective

- To recognise and draw the graphs of more complex linear equations

Key words
linear equation
variable

A **linear equation** connects two **variables** by a simple rule. Linear equations use any of the four operations: addition, subtraction, multiplication and division.

You have already met some examples of linear equations such as:

- $y = x + 2$
- $y = 3x$
- $y = 2x + 1$
- $y = 8 - x$

and you have drawn graphs to represent them. The two variables x and y are connected by a simple rule each time.

Example 1

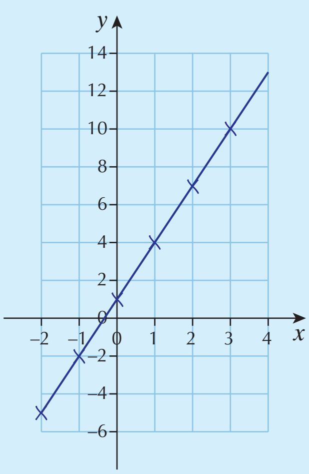
Draw a graph of the equation $y = 3x + 1$.

First, draw up a table of simple values for x. Then substitute that value of x in the equation to determine the corresponding y value.

x	−2	−1	0	1	2	3
$3x$	−6	−3	0	3	6	9
$y = 3x + 1$	−5	−2	1	4	7	10

The middle line of the table helps you to work towards the final value of $3x + 1$.

Finally, take the pairs of (x, y) coordinates from the table, plot each point on a grid, and join up all the points.



Notice that the line passes through other coordinates, as well. All of these fit the same equation, that is $y = 3x + 1$. Choose any points on the line that have not been plotted in the table and show that this is true.

Example 2

Draw a graph of the equation $2y + 3x = 6$.

To draw any straight line graph you need to find three suitable coordinates that fit the equation. Then draw a line to join up the points.

You could use two points but having a third point allows you to check that you have calculated the values and plotted them correctly.

The easiest points to find are where the line crosses the axes, that is when $x = 0$ and when $y = 0$. You also need to choose a third value. In this case use $x = 1$.

So, for $2y + 3x = 6$:

When $x = 0$, $2y = 6$, hence $y = 3$.

The coordinate is (0, 3).

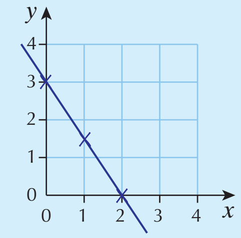
When $y = 0$, $3x = 6$, hence $x = 2$.

The coordinate is (2, 0).

When $x = 1$, $2y + 3 = 6$, hence $2y = 3$, and $y = 1.5$.

The coordinate is (1, 1.5).

Now you can plot each point and draw the graph as shown.



Notice that the third point (1,1.5) is on the line joining the first two points. It shows that you have drawn the graph correctly.

Exercise 7A

1 **a** Copy and complete the table for the equation $y = 2x - 3$.

x	–1	0	1	2	3	4
$2x$		0				
$y = 2x - 3$		–3				

b Draw a coordinate grid, numbering the x-axis from –1 to 4 and the y-axis from –6 to 6.

c Use values from the table to draw, on the grid, the graph of $y = 2x - 3$.

2 **a** Copy and complete the table for the equation $y = 4x - 2$.

x	–1	0	1	2
$4x$		0		
$y = 4x - 2$		–2		

b Draw a coordinate grid, numbering the x-axis from –1 to 2 and the y-axis from –7 to 7.

c Use values from the table to draw, on the grid, the graph of $y = 4x - 2$.

3 **a** Copy and complete the table for the equation $y = 8 - x$.

x	–2	0	2	4	6	8
$y = 8 - x$		8				

b Draw a coordinate grid, numbering the x-axis from –2 to 9 and the y-axis from –1 to 12.

c Use values from the table to draw, on the grid, the graph of $y = 8 - x$.

4 Complete suitable tables and draw graphs for each equation.

a $y = 2x - 1$ **b** $y = 5x - 3$ **c** $y = 10 - x$ **d** $y = 6x - 5$

5 **a** Draw a coordinate grid, numbering the x-axis from –3 to 3 and the y-axis from –12 to 12.

b On this grid draw lines that represent each of these equations.

i $y = x$ **ii** $y = 2x$ **iii** $y = 3x$ **iv** $y = 4x$

MR **c** Comment on what you notice about all these graphs.

PS **d** Predict where the line representing $y = \frac{1}{2}x$ will be.

6 **a** Draw a coordinate grid, numbering the x-axis from –1 to 6 and the y-axis from –1 to 6.

b On this grid, draw lines representing each of these equations.

i $y = 5 - x$ **ii** $y = 4 - x$ **iii** $y = 3 - x$ **iv** $y = 2 - x$

MR **c** What properties do you notice about each line?

MR **d** Use the properties you have noticed to draw the graphs of these equations.

i $y = 6 - x$ **ii** $y = 1 - x$

7 By finding three suitable points, draw the graph of $4y + 2x = 8$.

8 By finding three suitable points, draw the graph of $3y + 4x = 24$.

PS **9** Draw the graph of $5y + 3x = 15$.

Challenge: Sloping graphs

A Draw, on the same set of axes, the graphs of these equations.

- $y = 0.5x - 2$
- $y = 0.5x + 2$

B Now draw, without any further calculations, the graphs of these equations.

- $y = 0.5x - 1$
- $y = 0.5x + 3$

7.2 Gradient (steepness) of a straight line

Learning objectives

- To work out the gradient of a graph from a linear equation
- To work out an equation of the form $y = mx + c$ from its graph

Key words	
constant	gradient
y-intercept	

The **gradient** of a straight line is its steepness or slope. You can measure it by calculating the increase in value up the y-axis for an increase of one in value along the x-axis.

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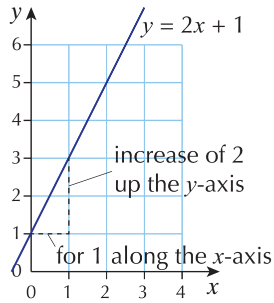
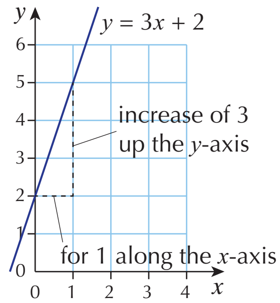
Here are some examples of gradients.

You will notice from these graphs that:

- the gradient is the same as the coefficient of x in the equation
- the line cuts the y-axis at the value that is added to the x-term.





The point where the line meets or cuts the y-axis is called the **y-intercept**.

A linear equation can be shown generally in the form

$$y = mx + c$$

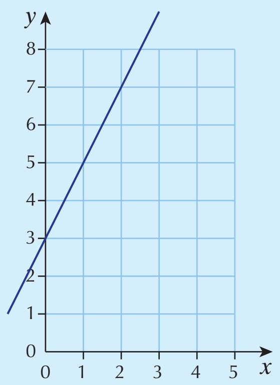
where m is the coefficient of x. The number added to x always stays the same so it is called the **constant** and is shown by c.

So, for any linear equation of the form $y = mx + c$:

- m is the same as the gradient of the line (steepness)
- c is the y-value where the line cuts through the y-axis.

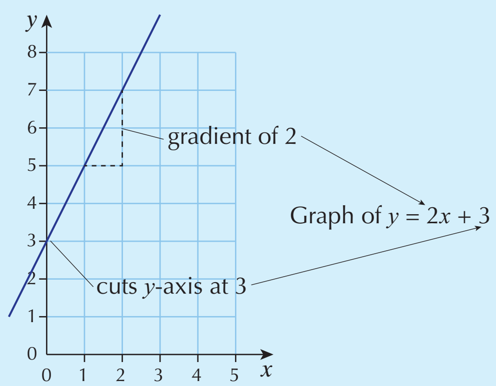
Example 3

What is the equation of this graph?



The gradient of the line is 2 and it cuts the y-axis at 3.

The equation of the graph is $y = 2x + 3$.



Example 4

Write down the equation of the straight line that cuts the y-axis at (0, 5) and has a gradient of 3.

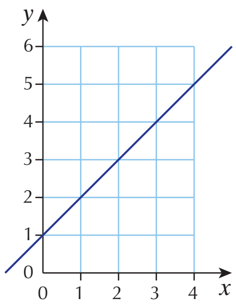
The equation of a straight line graph can be written as $y = mx + c$ where m is the gradient and c is the y-intercept.

So the equation of this line is $y = 3x + 5$.

Exercise 7B

1 State the gradient of each line.

a



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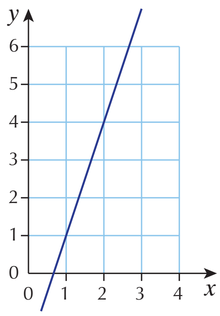
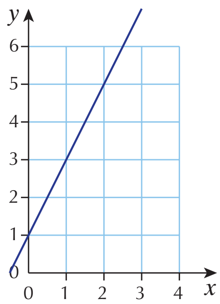
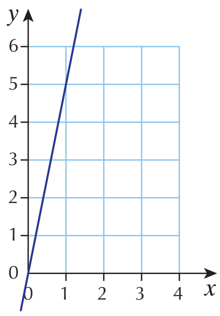


c



d



2 State the equation of the straight line with:

a a gradient of 4 passing through the y-axis at (0, 7)

b a gradient of 5 passing through the y-axis at (0, 1)

c a gradient of 6 passing through the y-axis at (0, 2)

d a gradient of 9 passing through the y-axis at (0, 13).

3 For each equation:

i write down the gradient

ii write down the intercept on the y-axis

iii sketch the graph without plotting points.

a $y = 2x + 1$ **b** $y = 3x - 4$ **c** $y = 4x + 1$

d $y = 5x - 3$ **e** $y = x + 2$ **f** $y = 0.5x + 7$

4 A straight line passes through the points (0, 1) and (3, 7).

a What are the coordinates of the point where this line cuts the y-axis?

b By plotting both points on a coordinate grid, calculate the gradient of the line.

c What is the equation of the line?

5 A straight line passes through the points (1, 2) and (2, 6).

a What are the coordinates of the point where this line cuts the y-axis?

b By plotting both points on a coordinate grid, calculate the gradient of the line.

c What is the equation of the line?

PS **6** What is the equation of the line that passes through each pair of coordinates?

a (0, 3) and (1, 6) **b** (0, 3) and (2, 7) **c** (2, 5) and (4, 9)

d (0, 2) and (1, 5) **e** (1, 3) and (2, 7) **f** (1, 1) and (4, 10)

Challenge: Different equations

Write down ten different equations with lines that pass through each point.

A (0, 5) **B** (2, 2) **C** (3, 7)

7.3 Graphs from quadratic equations

Learning objectives

- To recognise and draw the graph from a quadratic equation
- To solve a quadratic equation from a graph

Key word

quadratic

In a **quadratic** equation one of the variables is squared.

Here are some examples of simple quadratic equations.

- $y = x^2$
- $y = x^2 + 1$
- $y = 3x^2$
- $y = 2x^2 + 3x - 1$

You can follow the same technique of finding coordinates that fit the equation and plotting them on a graph. This time however the lines are not straight!

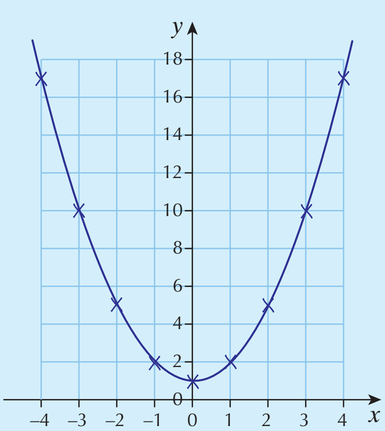
Example 5

Draw the graph of the equation $y = x^2 + 1$.

First, draw up a table of values for x. Then substitute each value of x into the equation to determine the corresponding y-value.

x	−4	−3	−2	−1	0	1	2	3	4
x^2	16	9	4	1	0	1	4	9	16
$y = x^2 + 1$	17	10	5	2	1	2	5	10	17

Now take the pairs of (x, y) coordinates from the table and plot each point on a coordinate grid. Join up all the points.



Notice the shape is a smooth curve. It is important to draw a quadratic graph as smoothly as possible, especially at the bottom of the graph where it needs to be a smooth curve.

Example 6

a Draw the graph of the equation $y = x^2 + x$.

b Use the graph to solve the equation $x^2 + x = 4$.

c What is the lowest possible value of y for this equation?

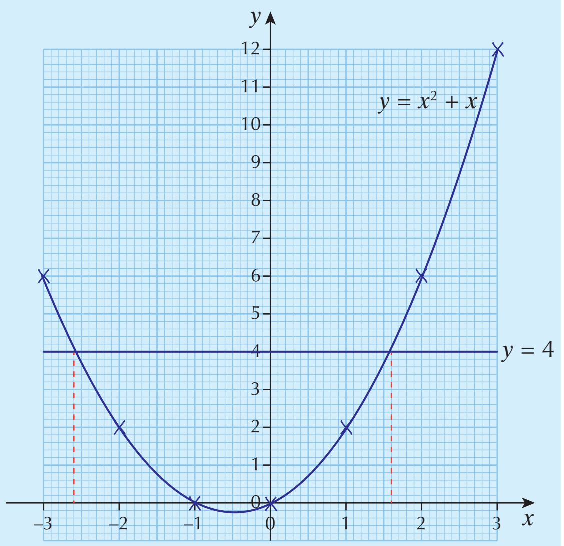
a Draw up a table of values for x. Then substitute each value of x to determine the corresponding y-value.

x	–3	–2	–1	0	1	2	3
x^2	9	4	1	0	1	4	9
$y = x^2 + x$	6	2	0	0	2	6	12

b To find the solution to $x^2 + x = 4$, draw the line $y = 4$ on the graph. Find the x-values of the points where the two lines intersect.

Follow the dashed lines on the diagram to find the x values at these points of intersection. They are $x = –2.6$ and $x = 1.6$.

c The smallest value of y for $y = x^2 + x$ is at the bottom of the curve below the y-axis. Read this value off the graph. The smallest value of $y = –0.2$.



Exercise 7C

1 **a** Copy and complete this table for the equation $y = x^2 + 5$.

x	–3	–2	–1	0	1	2	3
x^2	9	4	1	0	1	4	9
$y = x^2 + 5$				5			

b Draw a coordinate grid, numbering the x-axis from –3 to 3 and the y-axis from 0 to 15.

c Use values from the table to draw, on your grid, the graph of $y = x^2 + 5$.

2 **a** Copy and complete this table for the equation $y = x^2 + 4$.

x	–3	–2	–1	0	1	2	3
x^2							
$y = x^2 + 4$							

b Draw a coordinate grid, numbering the x-axis from –3 to 3 and the y-axis from 0 to 14.

c Use values from the table to draw, on your grid, the graph of $y = x^2 + 4$.

MR **3** Look at your answers to questions 1 and 2.

a What do you notice about each line?

b Use what you noticed to sketch the graphs of these two equations.

i $y = x^2 + 7$ **ii** $y = x^2 + 10$

c Explain what you expect to happen if you draw the graph for $y = x^2 - 5$.

d Draw the graph of $y = x^2 - 5$ and see if you were correct.

4 **a** Copy and complete this table for the equation $y = x^2 - x$.

x	–2	–1	0	1	2	3
x^2	4	1	0	1	4	9
$y = x^2 - x$						

b Draw a coordinate grid, numbering the x-axis from –2 to 3 and the y-axis from –1 to 8.

c Use values from the table to draw, on your grid, the graph of $y = x^2 - x$.

d Use your graph to solve the equation $x^2 - x = 5$.

e What is the smallest possible value of y in the equation $y = x^2 - x$?

5 **a** Copy and complete this table for the equation $y = 2x^2 + 3x$.

x	–3	–2	–1	0	1	2
$2x^2$	18					
$3x$	–9					
$y = 2x^2 + 3x$	9					

b Draw a coordinate grid, numbering the x-axis from –3 to 3 and the y-axis from –1 to 15.

c Use values from your table to draw, on your grid, the graph of $y = 2x^2 + 3x$.

d Use your graph to solve the equation $2x^2 + 3x = 1$.

e What is the smallest possible value of y in the equation $y = 2x^2 + 3x$?

6 **a** Draw a coordinate grid, numbering the x-axis from –3 to 3 and the y-axis from –1 to 22.

b Draw up a table of values for x and use these values to find the corresponding y-values for $y = 2x^2 - x$.

c Draw the graph of $y = 2x^2 - x$. Remember the graph should have a smooth curve at the bottom.

d Use your graph to solve the equation $2x^2 - x = 3$.

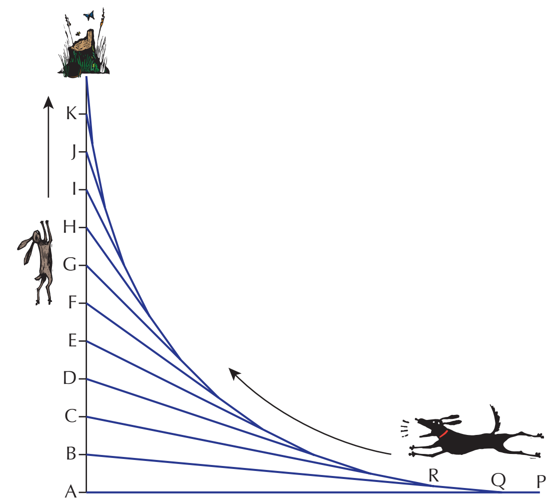
e What is the smallest possible value of y in the equation $y = 2x^2 - x$?

PS **7** Draw the graph of $y = 3x^2 + 4x$.

Challenge: The chase

A rabbit in a field suddenly saw a dog running straight towards it. The rabbit ran a straight line straight to its burrow, while the dog runs straight to the rabbit at all times.

The diagram shows the different directions the dog is running. For example, when the rabbit reaches point B, the dog is heading towards that point too, when the rabbit reaches point C, the dog is heading for point C too. The dog changes course as it sees the rabbit move from point to point. This process is repeated over and over until the rabbit reaches the hole.



The path traced out by the dog is called a tracetrix.

Draw the tracetrix for the dog in the diagram. Assume the dog runs at the same speed as the rabbit, so AB = BC = PQ = QR etc.

See if you can find out by how much the rabbit beats the dog to the hole.

7.4 Real-life graphs

Learning objective

- To draw graphs from real-life situations to illustrate the relationship between two variables

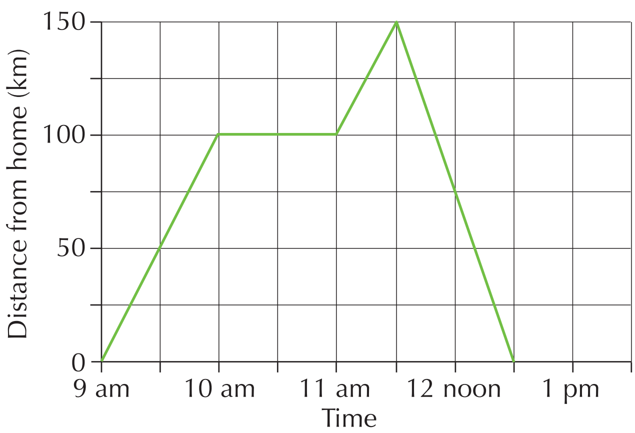
Key words

- average speed
- distance–time graph

You find graphs everywhere, in newspapers, advertisements, on TV and the internet.

Most of these graphs show a relationship between two variables. One variable is shown on one axis and the other is shown on the other axis.

You can use a **distance–time graph**, like the one here, to describe a journey.



The vertical axis shows distance travelled and the horizontal one shows time spent travelling.

This graph shows that:

- 100 km was travelled in the first hour
- then no distance at all was travelled for an hour (the line is flat)
- the journey continued for half an hour, from 11:00 am to 11:30 am
- the return journey started at 11:30 am.

Notice that the line shows the return journey by moving in the opposite direction. It goes down instead of up.

You can also use the graph to work out the **average speed** for a journey. In the example, you can see that the first 100 km were covered in 1 hour. This represents an average speed of 100 km/h. In general:

$$\text{average speed} = \frac{\text{total distance covered}}{\text{total time taken}}$$

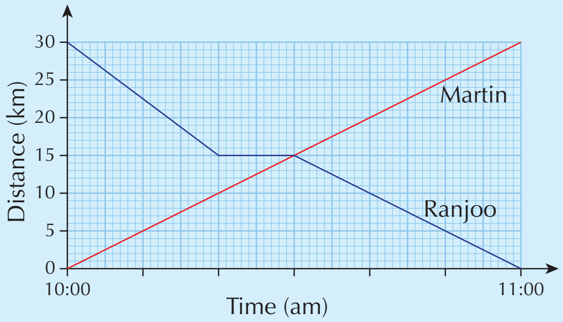
Example 7

Ranjoo and Martin live 30 km apart. Each person decided to cycle to the other's home. They did not notice each other as they passed and each ended up at the other's home.

The diagram shows a distance–time graph of their journeys.

a Describe each journey.

b At what time did they pass each other?



a Martin cycled 30 km in one hour. As the line is a straight line, he cycled at a steady speed. The speed was 30 km/h.

Ranjoo cycled 15 km in 20 minutes, stopped for 10 minutes, then finished his journey cycling 15 km in 30 minutes, at a speed of 30 km/h.

b They passed each other at 10:30 am.

Exercise 7D

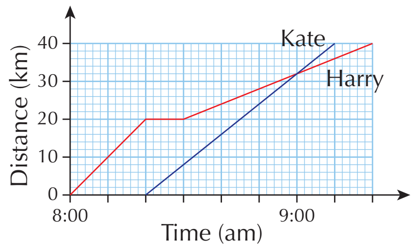
1 Harry and Kate were both travelling to Changhi Airport, but Kate drove at a faster speed.

The diagram represents both journeys.

a Describe Harry's journey, including the speed he travelled at.

b At what time did Kate overtake Harry?

c At what time did Harry get to the airport?



2 Jade sets out at 9 am on a 25 km walk. She walks at a steady speed of 6 km/h, but rests for 12 minutes after each hour's walk.

a Draw a distance–time graph to show her journey.

b At what time will Jade finish her walk?

PS **3** At 11 am Dave starts walking at 6 km/h to a station 8 km away. He waits at the station 15 minutes for a bike being brought to him on a train. He cycles back home at 16 km/h. When does he get home?

4 Two friends live 60 km apart. They arrange to meet at a café between their two homes.

Yifang leaves her home at 9 am and cycles for 90 minutes at a steady 16 km/h. She stops for 30 minutes at a shop then sets off at a steady 18 km/h.

Jason leaves his house at 9:15 am and cycles at a steady 14 km/h for the first hour, then stops for 30 minutes talking to a friend, before continuing at a speed of 16 km/h.

a Show both these journeys on the same distance–time graph.

PS **b** The friends arrive at the café at exactly the same time. What time do the friends meet?

c Who has cycled the furthest?

PS **5** Three cars start a race one after another at 5 minute intervals.

Car A starts first at 11 am and travels at a steady speed of 60 km/h.

Car B leaves next and travels at a steady speed of 90 km/h.

Car C leaves last and travels at a steady speed of 120 km/h.

At what time does:

a car B pass car A

b car C pass car A

c car C pass car B?

PS **6** A snail starts to climb a wall 4.5 metres high at 7 am and climbs 2 m every day. However the snail slips back down 50 cm every night. How long after the start will the snail reach the top of the wall?

Take day to start at 7 am and night to start at 7 pm.

Problem solving: Meeting in the middle

Two friends are walking towards each other on the same country path. Chris sets off at 9 am walking at 4 km/h. His dog, Alfie is with him. Lewis sets off 15 km away at 9:15 am, walking at 5 km/h. At 9:10 am, Alfie suddenly starts to run towards Lewis at 20 km/h, once there he turns round and runs back to Chris, then back to Lewis and so on until they all meet.

A What time do they all meet?

B How many times has Alfie run up towards Lewis?

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Ready to progress?

I can complete a table of values for a linear equation and use this to draw a graph of the equation.
I can calculate the gradient of a straight line drawn on a coordinate grid.
I can work out an equation of the form $y = mx + c$ from its graph.
I can draw and interpret graphs that illustrate real-life situations.

I can complete a table of values for a quadratic equation and use this to draw a graph of the equation.
I can solve a quadratic equation using a graph.

Review questions

1 Lynn went for a walk.

The distance–time graph shows information about her walk.

Which one of the statements below best describes her walk?

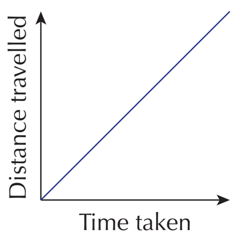
A She was walking faster and faster.

B She was walking slower and slower.

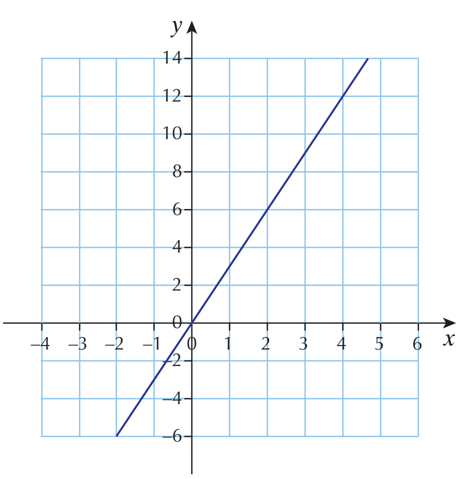
C She was walking north-east.

D She was walking at a steady speed.

E She was walking uphill.



MR **2** The graph shows the straight line with equation $y = 3x$.



a What is the gradient of this line?

b Does the point (25, 75) lie on the straight line $y = 3x$?
Explain how you know.

3 Dev uses a running machine to keep fit.

The simplified distance–time graph shows how he used the machine during one exercise routine.

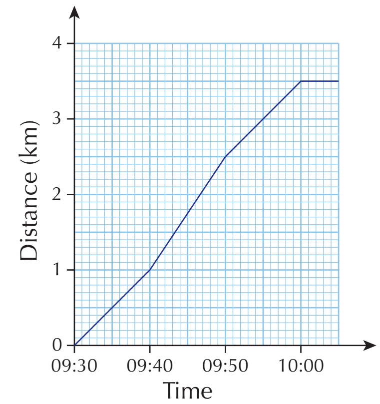
Use the graph to answer these questions.

a What was his speed in kilometres per hour between 09:30 and 09:40?

b How many minutes did he run at this speed for during the whole period of exercise?

c At 09:40, he increased his speed.

How many kilometres per hour did he increase his speed by?



MR **4** The graph shows the straight line with equation $y = 3x - 4$.

a A point on the line $y = 3x - 4$ has an x-coordinate of 50.

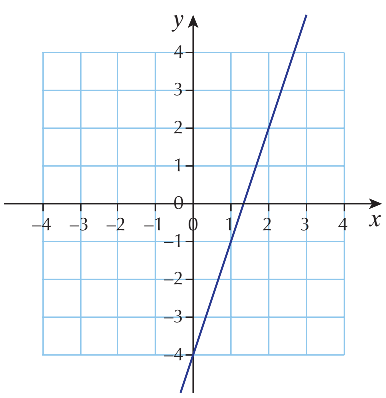
What is the y-coordinate of this point?

b A point on the line $y = 3x - 4$ has a y-coordinate of 50.

What is the x-coordinate of this point?

c Is the point (–10, –34) on the line $y = 3x - 4$?

Show how you know.



5 The graph shows a straight line with gradient 1.

a On a copy of the graph, draw a different straight line with gradient 1.

b The equation of another straight line is $y = 5x + 20$.

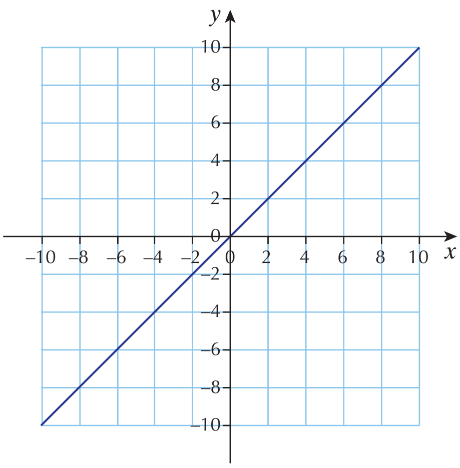
Write down the missing number.

The straight line $y = 5x + 20$ passes through $(0, \ldots)$.

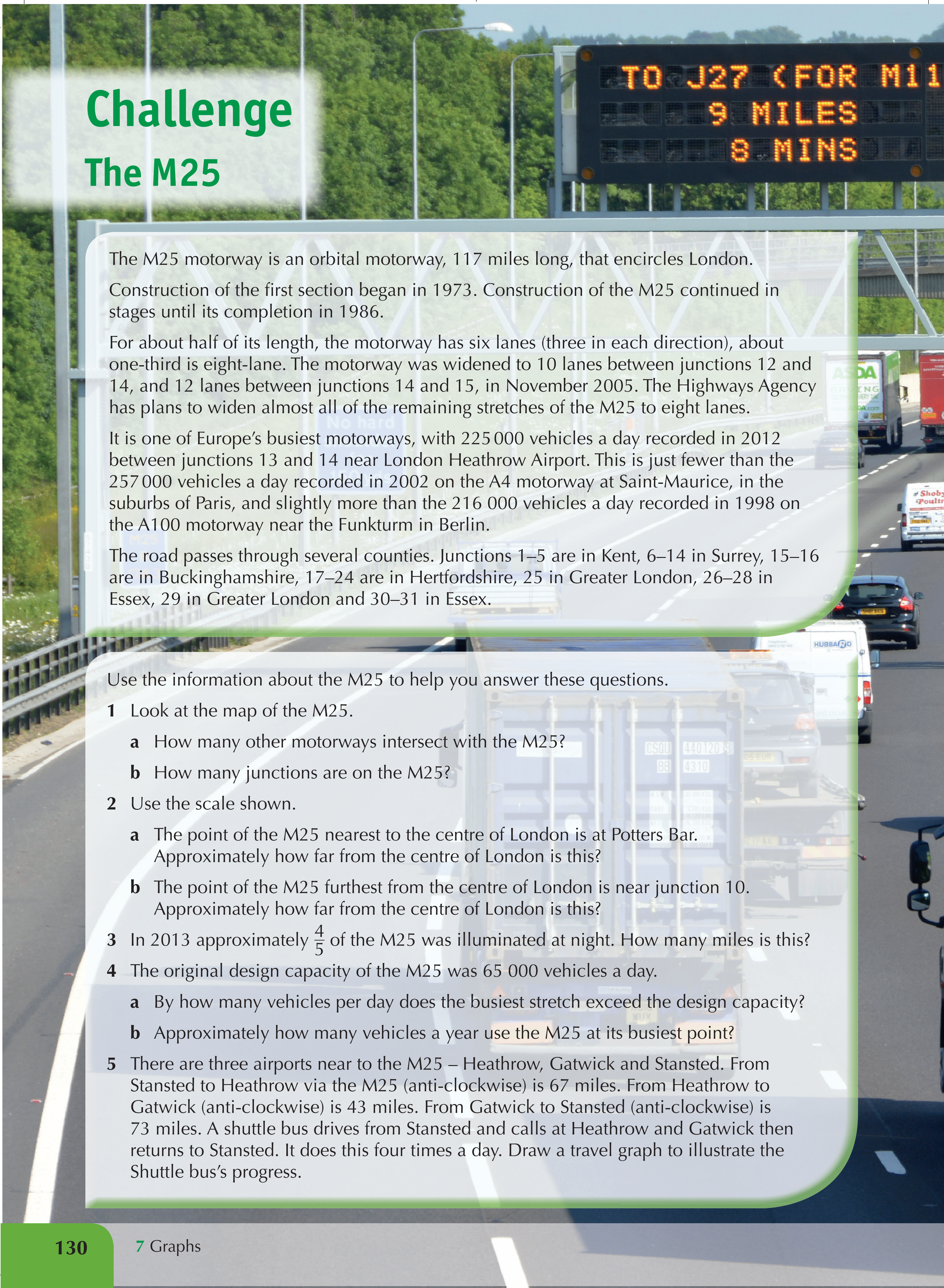
c A straight line is parallel to the line with equation $y = 5x + 20$.

It passes through the point (0, 10).

What is the equation of this straight line?



6 Draw the graph with the equation $y = x^2 - 3x$.



Challenge

The M25

The M25 motorway is an orbital motorway, 117 miles long, that encircles London.

Construction of the first section began in 1973. Construction of the M25 continued in stages until its completion in 1986.

For about half of its length, the motorway has six lanes (three in each direction), about one-third is eight-lane. The motorway was widened to 10 lanes between junctions 12 and 14, and 12 lanes between junctions 14 and 15, in November 2005. The Highways Agency has plans to widen almost all of the remaining stretches of the M25 to eight lanes.

It is one of Europe's busiest motorways, with 225 000 vehicles a day recorded in 2012 between junctions 13 and 14 near London Heathrow Airport. This is just fewer than the 257 000 vehicles a day recorded in 2002 on the A4 motorway at Saint-Maurice, in the suburbs of Paris, and slightly more than the 216 000 vehicles a day recorded in 1998 on the A100 motorway near the Funkturm in Berlin.

The road passes through several counties. Junctions 1–5 are in Kent, 6–14 in Surrey, 15–16 are in Buckinghamshire, 17–24 are in Hertfordshire, 25 in Greater London, 26–28 in Essex, 29 in Greater London and 30–31 in Essex.

Use the information about the M25 to help you answer these questions.

1 Look at the map of the M25.

- **a** How many other motorways intersect with the M25?
- **b** How many junctions are on the M25?

2 Use the scale shown.

- **a** The point of the M25 nearest to the centre of London is at Potters Bar. Approximately how far from the centre of London is this?
- **b** The point of the M25 furthest from the centre of London is near junction 10. Approximately how far from the centre of London is this?

3 In 2013 approximately $\frac{4}{5}$ of the M25 was illuminated at night. How many miles is this?

4 The original design capacity of the M25 was 65 000 vehicles a day.

- **a** By how many vehicles per day does the busiest stretch exceed the design capacity?
- **b** Approximately how many vehicles a year use the M25 at its busiest point?

5 There are three airports near to the M25 – Heathrow, Gatwick and Stansted. From Stansted to Heathrow via the M25 (anti-clockwise) is 67 miles. From Heathrow to Gatwick (anti-clockwise) is 43 miles. From Gatwick to Stansted (anti-clockwise) is 73 miles. A shuttle bus drives from Stansted and calls at Heathrow and Gatwick then returns to Stansted. It does this four times a day. Draw a travel graph to illustrate the Shuttle bus's progress.

6 The legal speed limit on the M25 is 70 mph. Assuming no hold ups how long would it take to drive around the M25 at the legal speed limit? Give your answer in hours and minutes.

Hint distance = speed × time

7 Five miles is approximately equal to eight kilometres. How long, to the nearest kilometre, is the M25?

8 The longest traffic jam on the M25 was 22 miles long. What percentage of the total length was the length of the jam?

9 This table shows the lengths of the three longest orbital motorways in the world and the length of the M60, which is the only other orbital motorway in Britain.

City	Country	Road	Length (miles)
Berlin	Germany	B-10	122
London	England	M25	117
Cincinnati	USA	I-275	84
Manchester	England	M60	35

The formula for the radius of a circle given the circumference is $r = \frac{C}{2\pi}$.

a Assuming the M25 is a circle with a circumference of 117 miles what would the radius be?

b Making the same assumption, calculate the radii of the other orbital roads.

10 Near to Heathrow Airport what is the average number of vehicles using the M25 per hour each day? Give your answer to the nearest 100 vehicles.