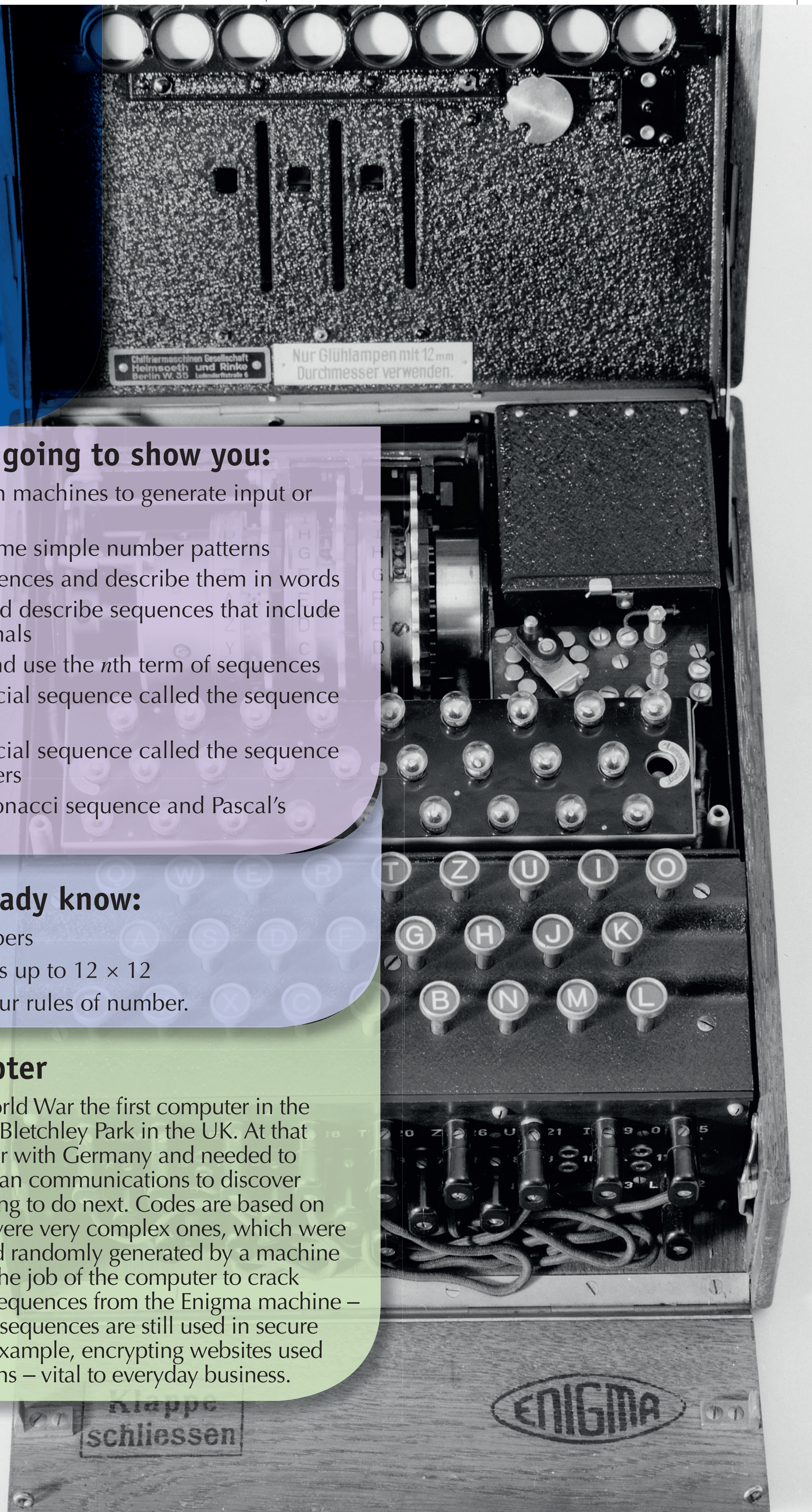
2

Sequences



This chapter is going to show you:

- how to use function machines to generate input or output values
- how to describe some simple number patterns
- how to create sequences and describe them in words
- how to generate and describe sequences that include fractions and decimals
- how to work out and use the *n*th term of sequences
- how to use the special sequence called the sequence of square numbers
- how to use the special sequence called the sequence of triangular numbers
- how to use the Fibonacci sequence and Pascal's triangle.

You should already know:

- odd and even numbers
- multiplication tables up to 12 × 12
- how to apply the four rules of number.

About this chapter

During the Second World War the first computer in the world was invented at Bletchley Park in the UK. At that time, Britain was at war with Germany and needed to break the coded German communications to discover what they were planning to do next. Codes are based on sequences and these were very complex ones, which were changed every day and randomly generated by a machine called Enigma. It was the job of the computer to crack each day's new code sequences from the Enigma machine – and fast. Today, coded sequences are still used in secure communications, for example, encrypting websites used for financial transactions – vital to everyday business.

2.1 Function machines

Learning objectives

- To use function machines to generate inputs and outputs
- To use given inputs and outputs to work out a function

Key words	
function	function machine
input	inverse
operation	output
rule	

In mathematics, a **function** is an **operation**, or a set of operations, that changes one number into another. There are many ways to show functions, one of which is the **function machine**. The numbers you start with are called the **input**. The numbers you get after you apply the operations are the **output**.

The operations in a function machine are called its **rules**.

Function machines are useful because they help you to understand the rules of algebra when you are solving equations.

Example 1

Complete a function machine to show the output for the numbers 3, 5 and 7 for the rule: multiply by 3 and then subtract 10.

→ Multiply by 3 → Subtract 10 →

Input	× 3		− 10	Output
3	→	9	→	−1
5	→	15	→	5
7	→	21	→	11

So the outputs are: −1, 5 and 11.

If you knew that the output from the function machine in Example 1 was 26, how could you work out the input?

You would work backwards by working out:

$$26 + 10 = 36$$

$$36 \div 3 = 12$$

So the input would be 12.

One way of doing this is to create an **inverse** function machine.

The inverse operation for add is subtract and vice versa and the inverse operation for multiplication is divide and vice versa.

The reverse function machine for Example 1 looks like this.

← Divide by 3 ← Add 10 ←

Example 2

a Draw an inverse function machine for this function machine.

→ Multiply by 5 → Subtract 8 →

b Work out the output for an input of 5 to the function machine.

c Put your answer for part **b** through the inverse function machine.

d Work out an input that gives the same value for the output.

a ← Divide by 5 ← Add 8 ←

b $5 \times 5 = 25$, $25 - 8 = 17$

c $17 + 8 = 25$, $25 \div 5 = 5$

d Guess an input and see if the outputs are the same. Then keep on guessing until you get the right answer. If the input is 2, then the output is $2 \times 5 = 10$, $10 - 8 = 2$.

This answer will work for the original function machine and the inverse function machine.

Exercise 2A

1 Write down the rule for each set of inputs and outputs, in words.

a

Input		Output
5	→	9
8	→	12
11	→	15
–3	→	1

b

Input		Output
6	→	18
8	→	24
10	→	30
21	→	63

c

Input		Output
$2\frac{1}{2}$	→	10
6	→	24
11	→	44
$12\frac{1}{2}$	→	50

d

Input		Output
3	→	$1\frac{1}{2}$
8	→	4
18	→	9
21	→	$10\frac{1}{2}$

e

Input		Output
5	→	–9
7	→	–7
9	→	–5
15	→	1

f

Input		Output
2	→	–2
4	→	–4
6	→	–6
8	→	–8

2 Work out the missing numbers in these function machines.

a → Multiply by 4 → Subtract … →

… → 20 → 11

b → Multiply by 5 → Add … →

… → 15 → 24

c → Add 4 → Multiply by … →

… → 7 → 21

d → Subtract 3 → Multiply by … →

… → $2\frac{1}{2}$ → 20

PS **3** Use inverse operations to work out the inputs for each of these function machines.

a

Input			Output
?	→ × 2 →	+ 3 →	7
?			13
?			16
?			25

b

Input			Output
?	→ − 5 →	÷ 2 →	14
?			16
?			20
?			26

c

Input			Output
?	→ × 3 →	+ 10 →	1
?			4
?			7
?			37

PS **4** The input and output for this function machine are both 10.

Work out possible values for the operations shown in the boxes.

10 → Multiply by … → Subtract … → 10

5 Each of these function machines uses two operations.

Work out what they are, for each one.

a

Input		Output
1	→	4
2	→	9
3	→	14
4	→	19

b

Input		Output
1	→	10
2	→	15
3	→	20
4	→	25

c

Input		Output
1	→	2
2	→	$2\frac{1}{2}$
3	→	3
4	→	$3\frac{1}{2}$

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6 Draw an inverse function machine for each of these function machines.
Use the inverse function machines to work out the input for an output of 42.

a → Multiply by 3 → Add 6 →

b → Add 9 → Multiply by 2 →

c → Divide by 2 → Add 36 →

d What do you notice about your answers?

PS **7** Work out the single input that gives the same output from both of these function machines.

a → Multply by 2 → Add 3 →

b → Multply by 5 → Subtract 6 →

PS **8** Work out the single input that gives the same output from both of these function machines.

a → Divide by 3 → Add 2 →

b → Add 6 → Multiply by 2 →

Challenge: Inputs and outputs

A Look at each number machine and work out the input that will give the same output as the input.

a → Multiply by 5 → Subtract 8 →

b → Multiply by 6 → Subtract 20 →

c → Divide by 2 → Subtract 3 →

d → Add 4 → Multiply by 2 →

e → Subtract 1 → Divide by 2 →

Hint **Answers may not be positive whole numbers.**

B These are the incomplete rules for a function machine.

→ Multiply by … → Add … →

a Work out numbers to go in the rule boxes so that:

i input 5 → output 17 **ii** input 8 → output 60 **iii** input 13 → output 65.

b Look back at part **a**. In each example you could put more than one pair of numbers into the rule boxes to give those inputs and outputs. See how many pairs you can find.

2.2 Sequences and rules

Learning objective

- To recognise, describe and generate sequences that follow a simple rule

Key words	
first term	geometric sequence
linear sequence	sequence
term	term-to-term rule

A **sequence** is a list of numbers that follow a pattern or rule. You can use simple rules to make up many different sequences with whole numbers. Sequences may also have different starting points. With *different* rules and *different* starting points, there are very many *different* sequences you may make.

Example 3

Rule: add 3

Starting at 1 gives the sequence 1, 4, 7, 10, 13, …
Starting at 2 gives the sequence 2, 5, 8, 11, 14, …
Starting at 6 gives the sequence 6, 9, 12, 15, 18, …

Example 4

Rule: double

Starting at 1 gives the sequence 1, 2, 4, 8, 16, …
Starting at 3 gives the sequence 3, 6, 12, 24, 48, …
Starting at 5 gives the sequence 5, 10, 20, 40, 80, …

The numbers in a sequence are called **terms** and the starting number is called the **first term**. The rule is often called the **term-to-term rule**.

Sequences that increase or decrease by a fixed amount from one term to the next are called **linear sequences**. Example 3 above shows linear sequences.

Sequences where each term after the first is found by multiplying or dividing by a fixed amount are called **geometric sequences**. Example 4 above shows geometric sequences.

Exercise 2B

1 Use each term-to-term rule to write down the first five terms of a sequence.
Start from a first term of 1.

a add 6 **b** multiply by 9 **c** subtract 3 **d** divide by 2

2 Use each term-to-term rule in question 1, starting from a first term of 5, to write down the first five terms of a new sequence.

3 Work out the next two terms in each sequence.
Describe the term-to-term rule you have used.

a 3, 7, 11, …, … **b** 4, 16, 64, …, … **c** 1, 5, 25, …, …
d 8, 16, 32, …, … **e** 100, 50, 25, …, … **f** 9, 5, 1, …, …
g 1000, 200, 40, …, … **h** 2, 6, 18, …, …

4 Work out the next two terms in each sequence.

Describe the term-to-term rule you have used.

a 20, 15, 10, 5, 0, …, …
b 4, 2, 1, $\frac{1}{2}$, $\frac{1}{4}$, …, …
c 66, 50, 34, 18, 2, …, …
d 1000, 100, 10, 1, 0.1, …, …

5 For each pair of numbers, work out at least two different sequences and write down the next two terms.

Describe the term-to-term rule you have used.

a 1, 5, …, …
b 3, 6, …, …
c 2, 8, …, …
d 3, 12, …, …
e 4, 2, …, …
f 45, 15, …, …

PS **6** Work out two terms between each pair of numbers, to form a sequence.

Describe the term-to-term rule you have used.

a 1, …, …, 10
b 18, …, …, 3
c 5, …, …, 625
d 4, …, …, $\frac{1}{16}$
e 160, …, …, 20
f 3, …, …, 81

MR **7** A snail is at the bottom of a well that is 12 feet deep. During the day the snail climbs 3 feet up the well, but during the night it slides 2 feet back down again.

How many days will it take the snail to reach the top of the well?

MR **8** Six fence posts are fixed, at equal distances apart, in a straight line. The distance between the two end posts is 12 feet.

What distance is each post in the fence from the next one?

FS **9** Laura saves money for 10 weeks, starting with £5 in the first week, £10 the second week, £15 the third week and so on.

Ed saves money for 10 weeks, starting with 50p the first week, £1 the second week, £2 the third week and so on.

Who has more money after 10 weeks, and by how much?

Investigation: Common terms

A **a** Write down the first 20 terms of the sequence that starts 5, 9, 13, 17, 21, … .

b Write down the first 20 terms of the sequence that starts 3, 8, 13, 18, 23, … .

B 13 is a common term in both sequences.

a What are the other common terms?

b Is there a rule to them?

Hint: You could use a spreadsheet to do this investigation.

C Here are another two sequences.

1, 5, 9, 13, 17, …

2, 8, 14, 20, 26, …

Are there any common terms in both sequences?

Explain your answer.

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2.3 Working out missing terms

Learning objective

- To work out missing terms in a sequence

The terms in a sequence are described as the first term, second term, third term, fourth term and so on. You need to know how to work out any term in a sequence.

Example 5

Look at this sequence:

7, 10, 13, 16, …

What is the 5th term? What is the 25th term? What is the 50th term?

You first need to know what the term-to-term rule is.

You can see that you add 3 from one term to the next.

This is called an 'add 3' rule.

	+ 3	+ 3	+ 3		
7	10	13	16	…	
1st	2nd	3rd	4th	5th	… 50th

To get to the fifth term, you add 3 to the fourth term, which gives 19.

To get to the 25th term, you will have to add on 3 a total of 24 times (25 – 1) to the first term, 7.

This will give $7 + 3 \times 24 = 7 + 72 = 79$.

To get to the 50th term, you will have to add on 3 a total of 49 times (50 – 1) to the first term, 7.

This will give $7 + 3 \times 49 = 7 + 147 = 154$.

Example 6

Look at this sequence:

45, 40, 35, 30, …

What is the 5th term? What is the 50th term?

	− 5	− 5	− 5		
45	40	35	30	…	
1st	2nd	3rd	4th	5th	… 50th

This is a 'subtract 5' rule. To get to the fifth term, you subtract 5 from the fourth term, which gives 25.

To get to the 50th term, you will have to subtract 5 a total of 49 times (50 – 1) from the first term, 45.

This will give $45 - 5 \times 49 = 45 - 245 = -200$.

Exercise 2C

1 Work out the fifth term and the 50th term in each sequence.

a 7, 13, 19, 25, … **b** 2, 7, 12, 17, … **c** 7, 14, 21, 28, …
d 10, 18, 26, 34, … **e** 1, 9, 17, 25, … **f** 17, 27, 37, 47, …

PS **2** In each sequence, you have been given the fourth, fifth and sixth terms.
Work out the first term and the 50th term in each case.

a …, …, …, 5, 7, 9, … **b** …, …, …, 7, 12, 17, …
c …, …, …, 9, 13, 17, … **d** …, …, …, 2, 11, 20, …

3 Work out the 20th term in each sequence.

a 98, 95, 92, 89, 86, …. **b** 57, 50, 43, 36, 29, ….
c 42, 38, 34, 30, 26, ….. **d** 38, 31, 24, 17, 10, …

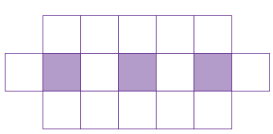
4 Work out the missing terms and the 50th term in each sequence.

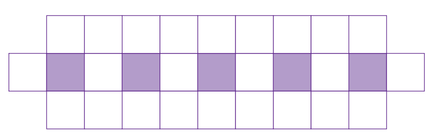
Term	1st	2nd	3rd	4th	5th	6th	7th	8th	50th
Sequence A	…	…	…	…	12	16	20	24	…
Sequence B	…	2	…	12	…	22	…	32	…
Sequence C	…	…	9	16	…	30	37	…	…
Sequence D	…	…	11	…	33	…	…	66	…

5 Work out the 30th term in the sequence with the term-to-term rule 'add 7' and a first term of 5.

6 Work out the 20th term in the sequence with the term-to-term rule 'subtract 5' and a first term of 94.

PS **7** These patterns are made from mauve and white squares.
The diagrams show the patterns for the third and fifth terms.


Pattern 3


Pattern 5

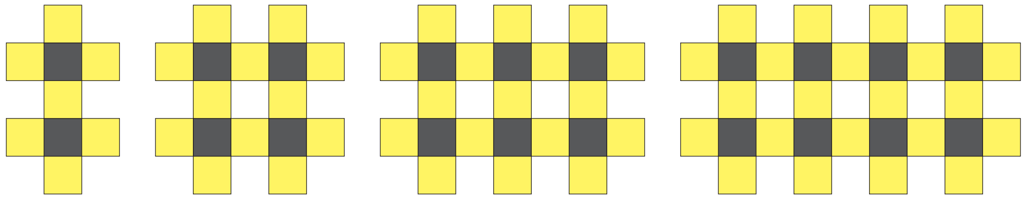
a How many mauve squares are there in the pattern for the fourth term?
b How many white squares are there in the pattern for the fourth term?
c Draw the pattern for the first term.
d How many squares in total are there in the pattern for the 10th term?

PS **8** The second and third terms of a sequence are 3 and 6.
…, 3, 6, …

There are several different sequences that could have 3 and 6 as the second and third terms.

a Write down one rule for the sequence and work out the first and fourth terms.
b Write down a different rule for the sequence and work out the first and fourth terms.

MR 9 You are using black and yellow squares to build patterns.



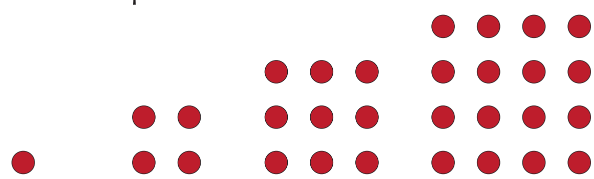
Pattern 1 Pattern 2 Pattern 3 Pattern 4

You have 50 black squares.

How many yellow squares will you need for that pattern?

Challenge: Patterns with dots

A Look at this pattern.

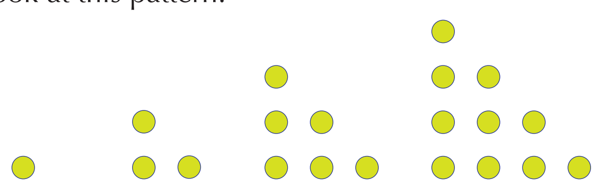


Pattern 1 Pattern 2 Pattern 3 Pattern 4

How many dots will there be in:

a the fifth pattern **b** the 10th pattern?

B Look at this pattern.



Pattern 1 Pattern 2 Pattern 3 Pattern 4

How many dots will be in:

a the fifth pattern **b** the 10th pattern?

C **a** Look again at the patterns in part **A**. Explain how you would work out that the number of dots in the 50th pattern is 2500.

b Look again at the patterns in part **B**. Explain how you would work out that the number of dots in the 50th pattern is 1275.

Hint **First double the number of dots in each pattern and look for a pattern.**

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2.4 Working out the *n*th term

Learning objectives

- To work out the *n*th term
- To use the *n*th term to work out any term in a sequence

Key words

algebraic rule	*n*th term

In any sequence, you can use the letter n to represent the number of the term in the sequence. Then the ***n*th term** of a sequence gives an **algebraic rule** for finding any term. You can also use the *n*th term to describe the sequence.

Example 7

What is the 10th term of this sequence?

2, 5, 8, 11, ….

This is an 'add 3' rule, so work out $2 + (10 - 1) \times 3 = 29$.

This is the same as $2 + (30 - 3) = 3 \times 10 - 1 = 29$.

You can use the same rule to work out the 20th term. This will be $3 \times 20 - 1 = 59$.

Now use the rule to work out the *n*th term.

Remember that the letter n stands for the number of any term in the sequence. So the *n*th term will be $3 \times n - 1$ which you write as $3n - 1$. This gives an algebraic rule to work out the *n*th term.

Now use this algebraic rule to work out the 100th term. So if $n = 100$ then $3n - 1 = 3 \times 100 - 1 = 299$.

Example 8

The *n*th term of the sequence 8, 13, 18, 23, 28, … is given by the algebraic rule $5n + 3$.

a Show that this is true for the first three terms.

b Use the rule to work out the 50th term of the sequence.

a When $n = 1$: $5 \times 1 + 3 = 5 + 3 = 8$

When $n = 2$: $5 \times 2 + 3 = 10 + 3 = 13$

When $n = 3$: $5 \times 3 + 3 = 15 + 3 = 18$

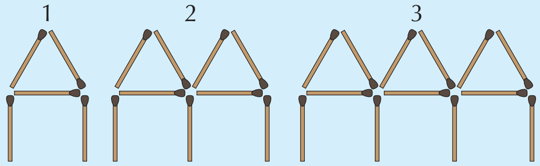
b When $n = 50$: $5 \times 50 + 3 = 250 + 3 = 253$

So the 50th term is 253.

Example 9

Look at the sequence of numbers based on this pattern of matchsticks.

Term number 1 2 3



Number of matchsticks 5 9 13

a Work out the *n*th term for the pattern.

b Work out the 50th term in this sequence.

a What is the term-to-term rule here? It is add 4, so the rule is based on $4n$.

The first term is 5.

For the first term, $n = 1$ and $4 \times 1 = 4$, but the first term is 5, so you need to add 1 to $4n$.

So the nth term is $4n + 1$.

b Now use this rule to work out the 50th term in the pattern.

When $n = 50$, $4n + 1 = 4 \times 50 + 1 = 201$.

Example 10

Work out the nth term of the sequence 2, 5, 8, 11, 14, …

This is an 'add 3' rule so the rule is based on $3n$.

The first term is 2, so subtract 1 from $3n$.

Hence the nth term is $3n - 1$.

Exercise 2D

1 Work out:

i the first three terms **ii** the 100th term

for the sequence, if the nth term is:

a $2n + 1$ **b** $4n - 1$ **c** $5n - 3$

d $3n + 2$ **e** $4n + 5$ **f** $10n + 1$

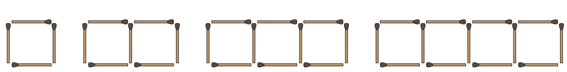
g $\frac{1}{2}n + 2$ **h** $7n - 1$ **i** $\frac{1}{2}n - \frac{1}{4}$.

2 For each of the patterns below, work out:

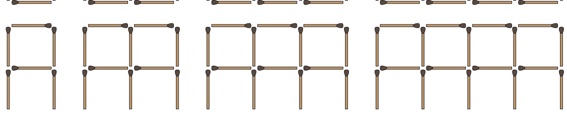
i the nth term for the number of matchsticks

ii the number of matchsticks in the 50th term.

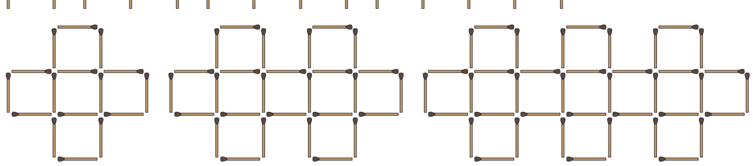
a



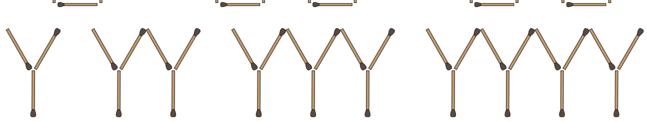
b



c



d



3 Work out the nth term of each sequence.

a 3, 9, 15, 21, 27, … **b** 10, 13, 16, 19, 22, … **c** 7, 13, 19, 25, 31, …

d 1, 4, 7, 10, 13, … **e** 4, 11, 18, 25, 32, … **f** 5, 7, 9, 11, 13, …

g 9, 13, 17, 21, 25, … **h** 5, 13, 21, 29, 37, … **i** 11, 21, 31, 41, 51, …

j 3, 12, 21, 30, 39, …

4 Each pattern below is made from matchsticks of two different colours.
Work out the nth term for:

i the number of red-tipped matchsticks

ii the number of blue-tipped matchsticks

iii the total number of matchsticks.

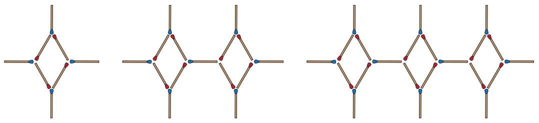
Use your rules to describe the 50th term in the patterns by working out:

iv the number of red-tipped matchsticks

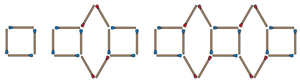
v the number of blue-tipped matchsticks

vi the total number of matchsticks.

a



b



5 Work out the nth term of each sequence.

a 95, 90, 85, 80, 75, …. **b** 33, 26, 19, 12, 5, …

c 24, 21, 18, 15, 12, … **d** 10, 8, 6, 4, 2 …

6 Work out the nth term of each sequence. They may include decimals or fractions.

a $2\frac{1}{2}, 4, 5\frac{1}{2}, 7, 8\frac{1}{2}, \ldots$ **b** 2.8, 4.1, 5.4, 6.7, 8.0, …

c $7, 9\frac{1}{2}, 12, 14\frac{1}{2}, 17, \ldots$ **d** 32.5, 30, 27.5, 25, 22.5, ….

7 Work out the first five terms of the sequence, if the nth term is:

a n^2 **b** $n^2 + 4$ **c** $n^2 - 1$

Hint **n^2 means work out $n \times n$.**

MR **8** What are the differences between the terms in each of the three sequences in question **7**? What do you notice?

Investigation: The eccentric mathematician

The daughter of a rich but eccentric mathematician had twin girls, Ellen and Emily.

The mathematician started a savings account for each of his granddaughters.

In one account he put in £5000 each year.

In the other account he put in £2 the first year, £4 the second year, £8 the third year and kept on putting in double the amount each year.

When the twins were 10 years old he asked them to choose which account they wanted.

A Show, by setting up a table or using a spreadsheet, that the first account had £50 000 in it and the second account had £2046.

B Ellen immediately chose the first account. Emily was left with the second account.
The grandfather kept on paying in money until the twins were 18 years old.
Which twin had more money in her account at age 18?

2.5 Other sequences

Learning objective

- To know and understand the square and triangular number sequences, the Fibonacci sequence and Pascal's triangle

Key words	
Fibonacci sequence	Pascal's triangle
power	square numbers
squaring	triangular numbers

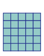
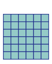
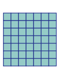
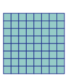
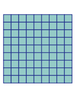
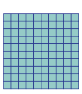
Square numbers and triangular numbers

When you multiply a number by itself you are **squaring** the number. The result is a **square number**. For example:

- 4 is a square number and it is the square of 2 ($2 \times 2 = 4$)
- 9 is a square number and it is the square of 3 ($3 \times 3 = 9$) and so on.

Instead of writing 1×1, 2×2, 3×3 and so on, you can write 1^2, 2^2, 3^2. This small 2 is called a **power** and, because it is so special, the power of 2 is also called 'square'. You refer to the number it is attached to as 'squared', for example, you say 5^2 as 'five squared'.

This table shows the first 10 square numbers. You can see from the bottom line of the table why they are called square numbers.

1×1	2×2	3×3	4×4	5×5	6×6	7×7	8×8	9×9	10×10
1^2	2^2	3^2	4^2	5^2	6^2	7^2	8^2	9^2	10^2
1	4	9	16	25	36	49	64	81	100
									

This is the start of the sequence of square numbers. You need to learn the square numbers up to $15^2 = 225$.

Another well-know sequence is 1, 3, 6, 10, 15, 21, ….

This is called the sequence of **triangular numbers**. This sequence builds up by adding on one more each time.

First term: 1

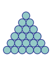
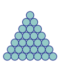
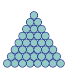
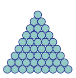
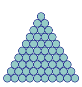
Second term: add 2 to the first term ($1 + 2 = 3$)

Third term: add 3 to the second term ($3 + 3 = 6$)

Fourth term: add 4 to the third term ($6 + 4 = 10$)

Fifth term: add 5 to the fourth term ($10 + 5 = 15$)

This table shows you the first 10 triangular numbers. You can see from the bottom line of the table why they are called triangular numbers.

1	1 + 2	3 + 3	6 + 4	10 + 5	15 + 6	21 + 7	28 + 8	36 + 9	45 + 10
1	3	6	10	15	21	28	36	45	55
									

You will find these numbers used in ten-pin bowling and snooker.

Exercise 2E

1 Copy the first three rows of the table of square numbers and continue it up to 15 × 15.

2 Write each number below as the sum of two square numbers. The first two have been done for you.

a 5 = 1 + 4	**b** 10 = 1 + 9	**c** 13 = … + …
d 17 = … + …	**e** 20 = ….. + …	**f** 25 = … + …
g 26 = … + …	**h** 29 = ….. + …	**i** 34 = … + …
j 37 = … + …	**k** 40 = ….. + …	**l** 41 = … + …
m 45 = … + …	**n** 50 = ….. + …	**o** 52 = … + …

3 Copy the first two rows of the table of triangular numbers and continue it up to the 15th triangular number.

4 Write each number as the sum of two triangular numbers. The first two have been done for you.

a 4 = 1 + 3	**b** 7 = 1 + 6	**c** 9 = … + …
d 11 = … + …	**e** 13 = … + …	**f** 16 = … + …
g 18 = … + …	**h** 21 = … + …	**i** 22 = … + …
j 24 = … + …	**k** 25 = … + …	**l** 27 = … + …
m 29 = … + …	**n** 31 = … + …	**o** 34 = … + …

5 Write down two numbers that are both square numbers and also triangular numbers.

6 Look at this pattern of numbers.

$$1 = 1 = 1^2$$
$$1 + 3 = 4 = 2^2$$
$$1 + 3 + 5 = 9 = 3^2$$
$$1 + 3 + 5 + 7 = 16 = 4^2$$
$$1 + 3 + 5 + 7 + 9 = 25 = 5^2$$

a Write down the next two lines of this number pattern.

b What is special about the numbers on the left hand side?

PS

c Without working them out, write down the answers to these calculations.

i 1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17 + 19 = …

ii 1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17 + 19 + 21 + 23 + 25 + 27 + 29 = …

7 Some sums of two square numbers are special because they give an answer that is also a square number. For example:

$3^2 + 4^2 = 9 + 16 = 25 = 5^2$

Which of these pairs of square numbers give a total that is also a square number?

a $5^2 + 12^2$	**b** $2^2 + 5^2$	**c** $6^2 + 8^2$
d $7^2 + 9^2$	**e** $7^2 + 24^2$	**f** $10^2 + 24^2$

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8 **a** Add up the first 10 pairs of consecutive square numbers, starting with $1 + 4$, $4 + 9$, $9 + 16$, … .

b Is it possible to get an even total if you add any pair of consecutive square numbers? If not, explain why not.

c Work out the differences between the totals.

Do they form a sequence? If so, describe the sequence.

9 Look at this pattern of numbers.

$$1 = 1$$
$$1 + 2 = 3$$
$$1 + 2 + 3 = 6$$
$$1 + 2 + 3 + 4 = 10$$
$$1 + 2 + 3 + 4 + 5 = 15$$

a Write down the next two lines of this number pattern.

b What is special about the numbers on the left-hand side?

c What is special about the numbers on the right-hand side?

PS **d** Without working them out, write down the answers to these calculations.

i $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 = \ldots$

ii $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 11 + 12 + 13 + 14 + 15 = \ldots$

10 **a** Add up the first 10 pairs of consecutive triangular numbers, starting with $1 + 3$, $3 + 6$, $6 + 10$, ….

b What is special about the answers?

Challenge: Testing rules

A Here is a rule.

$8 \times$ any triangular number $+ 1$ is a square number.

Test this rule for the first triangular number 1.

$8 \times 1 + 1 = 8 + 1 = 9 = 3^2$

Does the rule always work?

Test it using at least four other triangular numbers.

B Now test this rule.

$9 \times$ any triangular number $+ 1$ is also a triangular number.

C Now test this rule.

The sum of the squares of any two consecutive triangular numbers is also a triangular number.

Special sequences

The next two number patterns are not linear or geometric sequences but they are very important in mathematics.

The Fibonacci sequence can also be found in many patterns in nature.

The Fibonacci sequence

An Italian mathematician, Leonardo Fibonacci first wrote about this sequence in the 13th century. The sequence is:

1 1 2 3 5 8 13 21 …

Each term after the first two is the sum of the previous two terms.

The next term would be 13 + 21 = 34, and so on.

Pascal's triangle

A French mathematician, Blaise Pascal, first wrote about this number pattern, which is now known as Pascal's triangle.

Row													
1							1						
2						1		1					
3					1		2		1				
4				1		3		3		1			
5			1		4		6		4		1		
6		1		5		10		10		5		1	
7	1		6		15		20		15		6		1

Each row starts and finishes with 1.

Each of the other numbers is the sum of the two numbers above it, to the left and right.

For example, in row 5:

4 = 1 + 3   6 = 3 + 3

and in row 7:

15 = 10 + 5.

Exercise 2F

1. Continue the Fibonacci sequence until the 20th term.
2. Write down four numbers that are in both the sequence of triangular numbers and the Fibonacci sequence.
3. Write down two numbers that are in both the sequence of square numbers and the Fibonacci sequence.
4. Look again at Pascal's triangle, above, and write down the next three rows.

MR

5. Add up the numbers in each row of Pascal's triangle:

   1 = 1, 1 + 1 = 2, 1 + 2 + 1 = 4, …

   Describe the number pattern formed by the totals.

MR 6 Look again at Pascal's triangle.

a Find a diagonal that gives the counting numbers (1, 2, 3, 4, …).

b Find a diagonal that gives the triangular numbers.

MR 7 Add up the first three numbers in the Fibonacci sequence, then the first four, then the first five, and so on. Can you see any pattern emerging in the results?

MR 8 Write down any four consecutive numbers from the Fibonacci sequence, for example, 5, 8, 13, 21.

Multiply the first and the last together.

Multiply the middle two together.

What is the difference between the numbers?

Try this for at least another two sets of four consecutive Fibonacci numbers.

Does this rule work for any four consecutive Fibonacci numbers?

Investigation: Consecutive numbers

A Take any three consecutive numbers, for example, 7, 8, 9.

Multiply the first and last of your numbers. $7 \times 9 = 63$

Square the middle number. $8^2 = 64$

Subtract the first number from the second. $64 - 63 = 1$

Try this with at least three different sets of consecutive numbers.

What happens in every case?

B Now try the same process as in part **A** with any three consecutive Fibonacci numbers, For example 5, 8, 13.

Repeat this with at least three different sets of consecutive Fibonacci numbers.

What happens this time?

C Now try the same process as in part **A** with any three alternate consecutive Fibonacci numbers, for example, 3, 8, 21.

Repeat this with at least three different sets of alternate consecutive Fibonacci numbers.

What happens this time?

D Now try the same process as in part **A** with any three Fibonacci numbers that are consecutive but with a gap of two numbers between them, for example, 2, 8, 34.

Repeat this with at least three different similar sets of Fibonacci numbers.

What happens this time? Can you write down a rule?

Ready to progress?

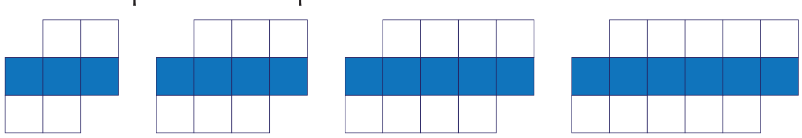
I can work out the term-to-term rule for a sequence.
I can work out the operation in a function machine that uses one rule, when I am given the inputs and outputs.
I know the square numbers up to 15×15.

I can work out the inputs when given the outputs and the rules for an inverse function machine.
I can work out any term in a sequence, given the first term and the term-to-term rule.
I can recognise and work out the sequence of triangular numbers.
I can investigate the patterns and connections within the square and triangular numbers.
I can recognise and work out the numbers in the Fibonacci sequence and in Pascal's triangle.
I can work out any term in a sequence, given the nth term.

I can work out the operations in a function machine that uses more than one rule, when I am given the input and output values.
I can work out the nth term for any linear sequence.
I can investigate a given rule and reach a conclusion about whether it always works.

Review questions

1 Here is a sequence of shapes made with blue and white tiles.



Shape 1 Shape 2 Shape 3 Shape 4

The number of blue tiles = the shape number + 2
The number of white tiles = $2 \times$ the shape number + 2

a How many blue tiles will there be in shape number 8?
b How many white tiles will there be in shape number 35?
c How many tiles altogether will there be in shape number 50?
d Write down the missing numbers from this sentence:
The total number of tiles = … × the shape number + …

2 In each of these function machines, the two rules can be replaced with one rule.
Work out what this is for each machine. The first one has been done for you.

a → + 2 → − 7 → = → − 5 →
b → − 7 → + 3 → = → ? →
c → × 9 → × 2 → = → ? →
d → × 4 → ÷ 8 → = → ? →

3 **a** Write down the next two numbers in the sequence below.

93, 85, 77, 69, 61, …, …

b Write down the next two numbers in this sequence.

1, 2, 4, 7, 11, 16, …, …

c What two numbers do both sequences have in common, if they are continued?

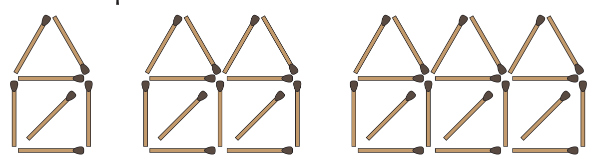
4 **a** Maisy saves £5 each week for 10 weeks. How much will she have after five weeks?

b Daisy saves £1 the first week, £2 the second week, £3 the third week, £4 the fourth week, and so on for 10 weeks. Copy and complete the table.

Week	1	2	3	4	5	6
Amount saved	£1	£2	£3	£4	£5	£6
Total amount saved	£1	£3				

c Work out who will have more money at the end of 10 weeks.

5 This is a sequence of huts made from matches.



1 hut
7 matches

2 huts
13 matches

3 huts
19 matches

a Work out the rule for the nth term (the number of matches in the nth hut).

b Work out how many matches will be needed for the 30th hut.

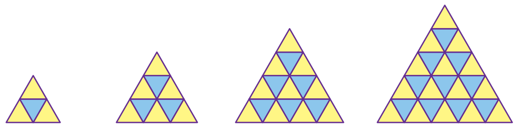
c I use 91 matches to make a sequence of huts. How many huts do I make?

d I have 200 matches. What is the largest number of huts I can make?
How many matches will be left over?

6 Work out the nth term for each sequence.

a 5, 14, 23, 32, 41, …, … **b** 12, 10, 8, 6, 4, …, … **c** 3.5, 4, 4.5, 5, 5.5, …, …

7 The patterns in this sequence are made from blue and yellow triangles.



Pattern 1 Pattern 2 Pattern 3 Pattern 4

a Copy and complete this table.

Pattern (term) number	Number of blue triangles	Number of yellow triangles	Total number of triangles
1	1	3	4
2	3		
3			
4			

b Describe the sequence formed by the numbers of blue triangles.

c Describe the sequence formed by the numbers of yellow triangles.

d Describe the sequence formed by the total numbers of triangles.

e Work out the total number of triangles in Pattern 10.

f What is the total number of triangles in the nth pattern?

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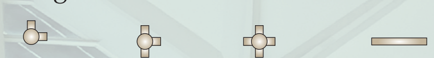
Mathematical reasoning

Valencia Planetarium

This is the planetarium in the city of Valencia in Spain. Many of the features of the building are based on a repeating sequence.

This key explains the diagrams on these pages.

Ladders and grids are made from combinations of:

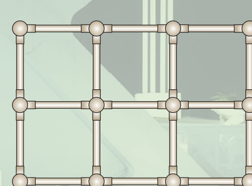


Each combination can be expressed as a rule, using letters.

Example

This grid is a combination of L links, T links, X links and R rods.

The combination can be written as the rule:
$4L + 6T + 2X + 17R$



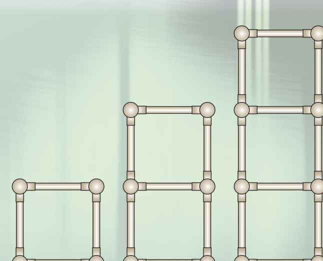
$4L + 6T + 2X + 17R$

1 Look at the ladders on the right.

a Use letters to write down a rule for each of them.

b Copy and complete this table.

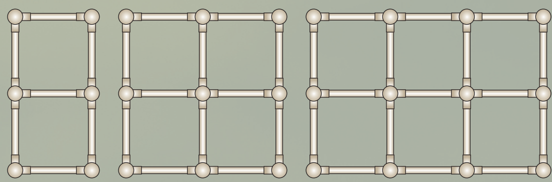
Ladder	L links	T links	R rods
1	4	0	4
2	4	2	7
3			
4			
5			



c Use letters to write down a rule for the links and rods in ladder 10.

d Use letters to write down a rule for the links and rods in ladder n.

2 These rectangles are 2 squares deep.



i ii iii

a Use letters to write down a rule for each of them.

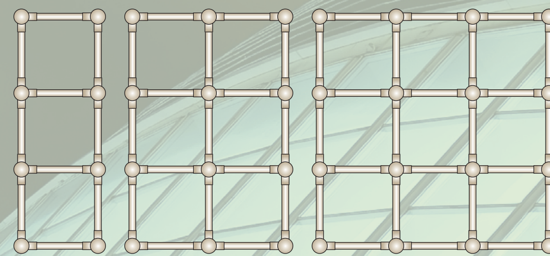
b Copy and complete this table.

Rectangle	L links	T links	X links	R rods
2 by 1	4	2	0	7
2 by 2	4	4	1	12
2 by 3				
2 by 4				
2 by 5				

c Use letters to write down a rule for the links and rods in a 2 by 10 rectangle.

d Use letters to write down a rule for the links and rods in a 2 by *n* rectangle.

3 These rectangles are 3 squares deep.



i ii iii

a Use letters to write down a rule for each of them.

b Copy and complete this table.

Rectangle	L links	T links	X links	R rods
3 by 1	4	4	0	10
3 by 2	4	6	2	17
3 by 3				
3 by 4				
3 by 5				

c Use letters to write down a rule for the links and rods in a 3 by 10 rectangle.

d Use letters to write down a rule for the links and rods in a 3 by *n* rectangle.

e Now write down a rule for the number of links and rods in a 4 by *n* rectangle.

4 a Use letters to write down a rule for each of these squares.

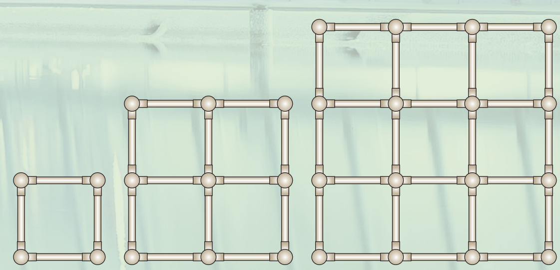
b Copy and complete this table.

Square	L links	T links	X links	R rods
1 by 1	4	0	0	4
2 by 2	4	4	1	12
3 by 3				

c Continue the table to work out the number of links and rods in a 7 by 7 square.

d Describe how the pattern of T links is building up.

e What type of special sequence do the numbers of X links make?



i ii iii

f Show that the number of rods in an *n* by *n* square is given by the *n*th term $2n(n + 1)$.

g Use letters to write down a rule for the number of links and rods in an 8 by 8 square.

h Use letters to write down a rule for the number of links and rods in an *n* by *n* square.

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