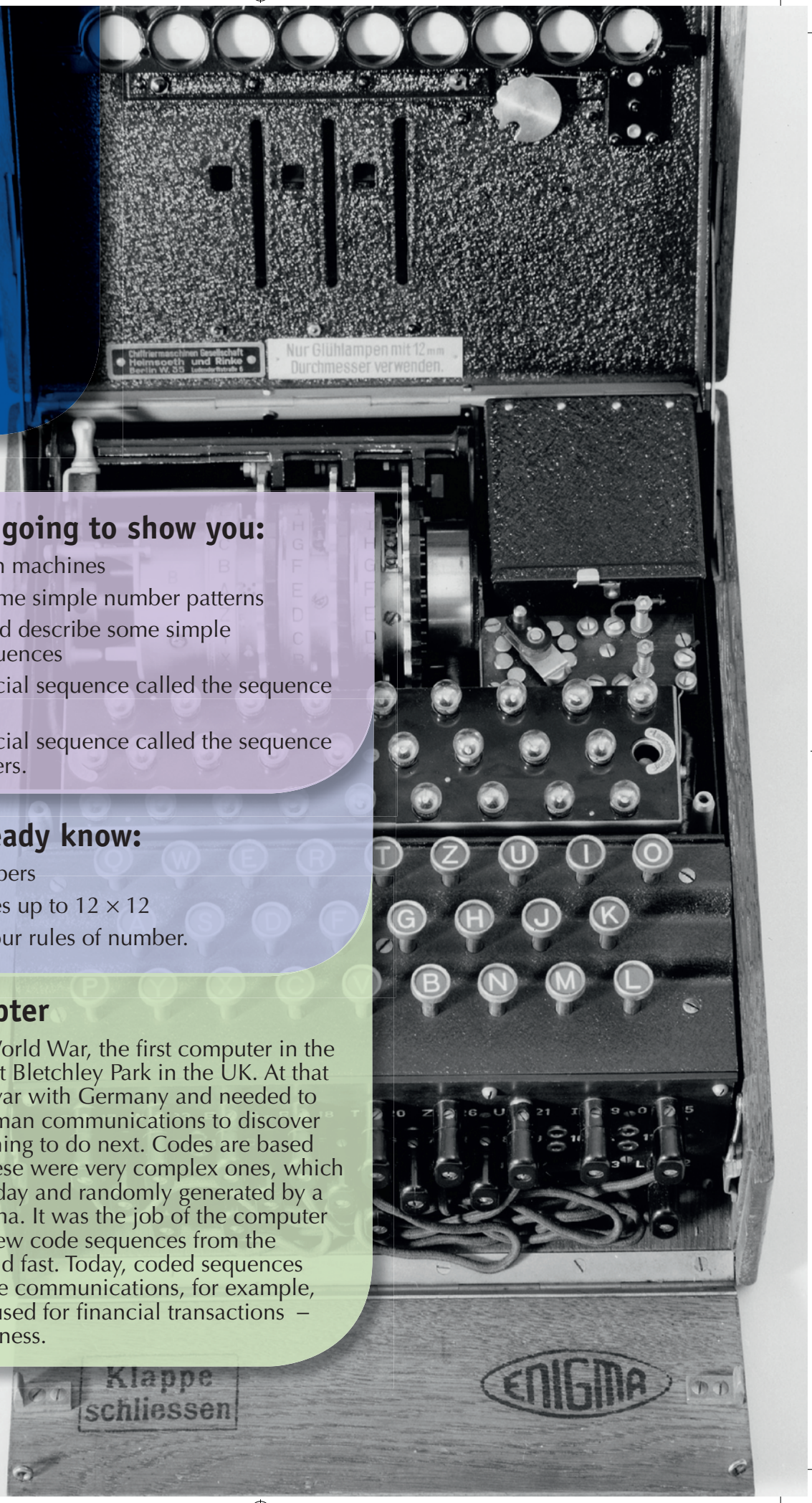
2 Sequences



This chapter is going to show you:

- how to use function machines
- how to describe some simple number patterns
- how to generate and describe some simple whole-number sequences
- how to use the special sequence called the sequence of square numbers
- how to use the special sequence called the sequence of triangular numbers.

You should already know:

- odd and even numbers
- multiplication tables up to 12×12
- how to apply the four rules of number.

About this chapter

During the Second World War, the first computer in the world was invented at Bletchley Park in the UK. At that time, Britain was at war with Germany and needed to break the coded German communications to discover what they were planning to do next. Codes are based on sequences and these were very complex ones, which were changed every day and randomly generated by a machine called Enigma. It was the job of the computer to crack each day's new code sequences from the Enigma machine – and fast. Today, coded sequences are still used in secure communications, for example, encrypting websites used for financial transactions – vital to everyday business.

2.1 Function machines

Learning objective

- To use function machines to generate inputs and outputs

Key words

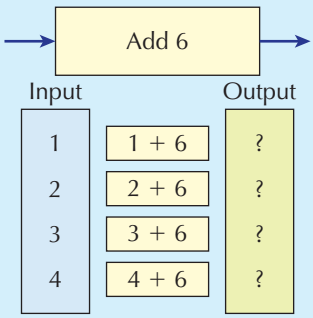
- double function machine
- function machine
- input
- inverse operation
- output

A **function machine** uses a mathematical rule to change the values of numbers.

- The numbers you start with are called the **input**.
- The numbers you get after you apply the rule are the **output**.

Example 1

Complete the output box of this function machine for inputs of 1, 2, 3 and 4.



The rule is 'Add 6'.

The first input is 1.

$1 + 6 = 7$

The first output is 7.

$2 + 6 = 8$

The second output is 8.

The third input is 3.

$3 + 6 = 9$

The third output is 9.

The fourth input is 4.

$4 + 6 = 10$

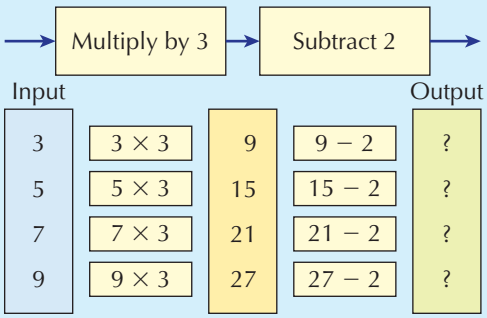
The fourth output is 10.

The output box looks like this.

Output
7
8
9
10

Example 2

Complete the **double function machine** to show the outputs for inputs of 3, 5, 7 and 9.



The rule is 'Multiply by 3' then 'Subtract 2'.

The first input is 3.

$3 \times 3 = 9$ and $9 - 2 = 7$

The first output is 7.

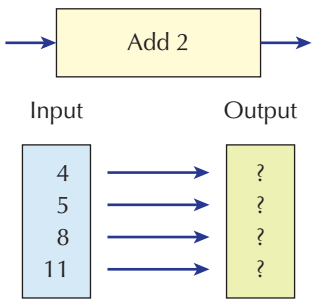
Continue in this way.

The output box looks like this.

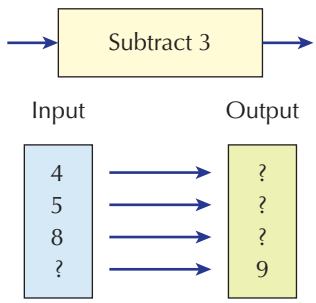
Exercise 2A

1 Work out the missing inputs or outputs for each function machine.

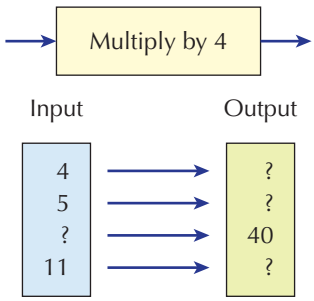
a



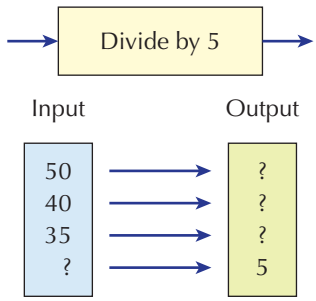
b



c

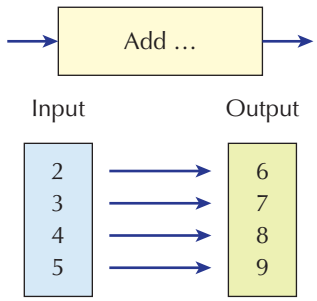


d

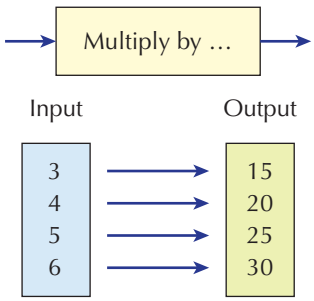


2 Complete the rule for each function machine.

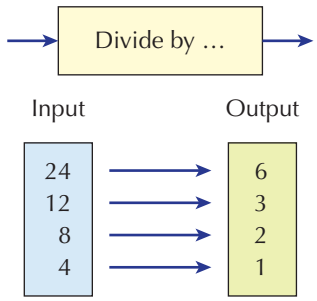
a



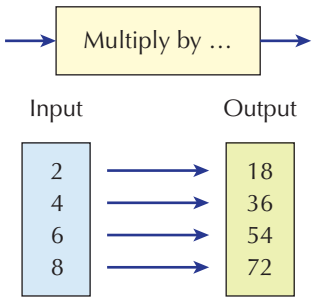
b



c

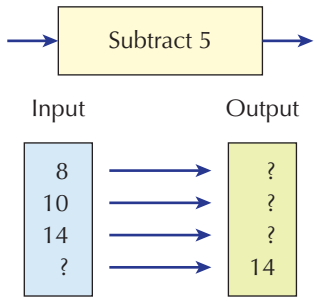


d

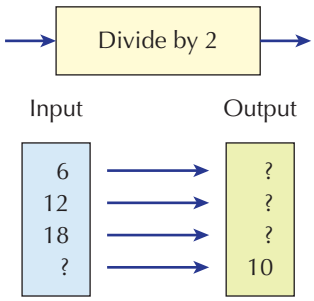


3 Work out the missing inputs or outputs for each function machine.

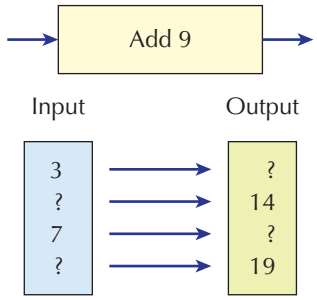
a



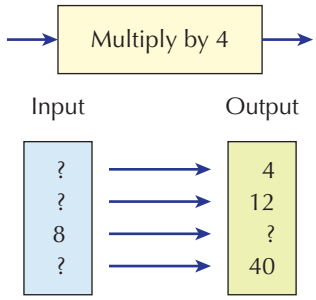
b



c



d



4 Make up your own function machines to show each of these functions.

Choose your four inputs for each one carefully, so that the outputs are not fractions or negative numbers.

a Add 1

b Multiply by 10

c Subtract 3

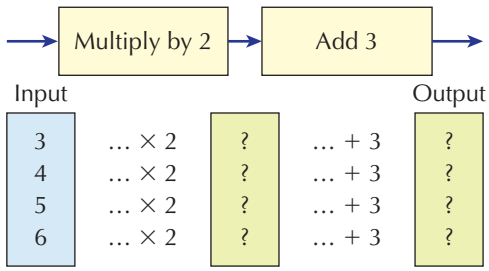
d Halve

e Add 11

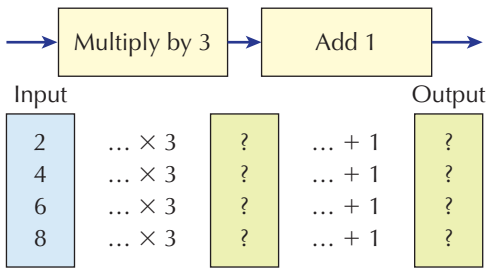
f Subtract 20

5 Work out the outputs for each of these double function machines.

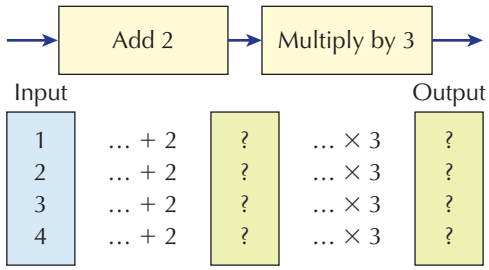
a



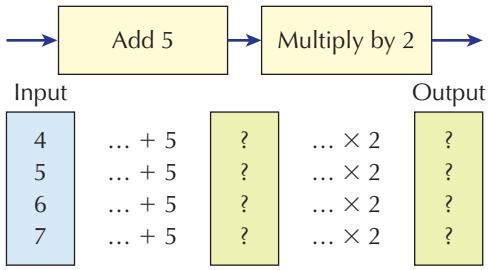
b



c



d



6 Draw diagrams to show each of these double function machines.
Choose your own four input numbers.

a Multiply by 3 → Subtract 2

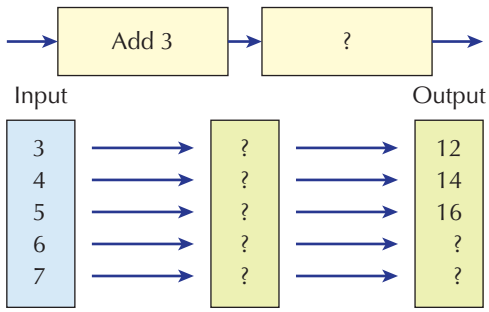
b Multiply by 5 → Subtract 4

c Subtract 3 → Multiply by 10

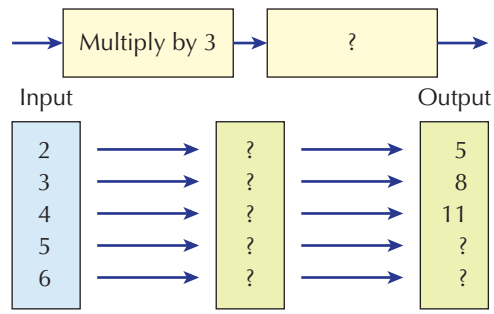
d Double → Add 8

7 Copy each double function machine. Fill in the missing rules and numbers.

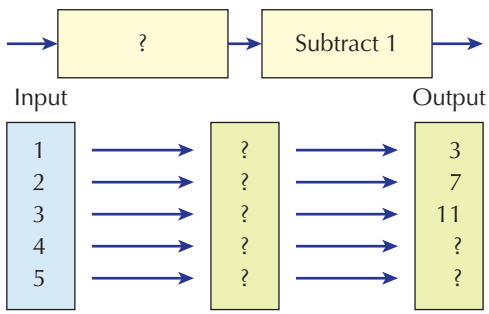
a



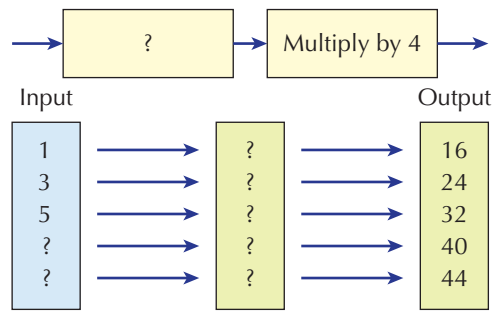
b



c



d



PS **8** Use **inverse operations** from each output to work out the input for each double function machine.

a ?, ?, ?, ? → × 2 → + 3 → 7, 13, 19, 25

b ?, ?, ?, ? → + 3 → × 2 → 14, 16, 20, 26

c ?, ?, ?, ? → × 4 → − 7 → 9, 17, 33, 37

Hint

Remember that using inverse operations means working backwards and reversing all the operations.

9 Each of these double function machines has been replaced with a single function machine.

Work out the operation that is missing from each double function machine.

a … → Add 2 → = → Add 6

b Add 2 → … → = → Subtract 5

c … → Multiply by 3 → = → Multiply by 9

d Multiply by 4 → … → = → Divide by 2

Challenge: Double function machines

The outputs come from using two rules on the inputs, as in Question 5 above.

Draw a double function machine for each one.

A

Input	Output
1	3
2	5
3	7
4	9

B

Input	Output
1	2
2	5
3	8
4	11

C

Input	Output
1	7
2	12
3	17
4	22

D

Input	Output
1	8
2	10
3	12
4	14

E

Input	Output
1	6
2	9
3	12
4	15

F

Input	Output
1	15
2	20
3	25
4	30

2.2 Sequences and rules

Learning objective

- To recognise, describe and write down sequences that are based on a simple rule

Key words	
first term	rule
sequence	term
term-to-term rule	

A **sequence** is a list of numbers that follow a set **rule**.

You can make up many different sequences with whole numbers, based on simple rules.

The different numbers in a sequence are called **terms**. The starting number is called the **first term**. The rule is called the **term-to-term rule.**

Example 3

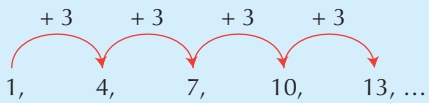
Look at the rule and the starting number, then write down the first five terms of the sequence.

a Rule: Add 3 **i** Starting number 1 **ii** Starting number 2

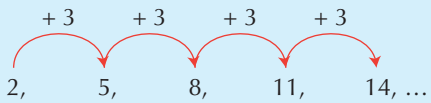
b Rule: Double **i** Starting number 1 **ii** Starting number 3

a Add 3

i Starting at 1 gives the sequence 1, 4, 7, 10, 13, ...

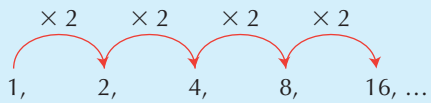


ii Starting at 2 gives a different sequence: 2, 5, 8, 11, 14, ...

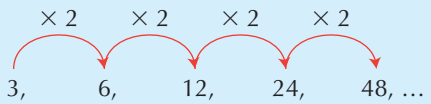


b Double

i Starting at 1 gives the sequence 1, 2, 4, 8, 16, ...



ii Starting at 3 gives a different sequence: 3, 6, 12, 24, 48, ...



With different rules and different starting points, you can make many different sequences.

Exercise 2B

1 Use each term-to-term rule and the given starting point to make a sequence with four terms in it.

a Rule: Add 3 Start at 2.
b Rule: Multiply by 3 Start at 1.
c Rule: Add 5 Start at 4.
d Rule: Multiply by 10 Start at 3.
e Rule: Add 9 Start at 6.
f Rule: Multiply by 5 Start at 2.
g Rule: Add 7 Start at 3.
h Rule: Multiply by 2 Start at 5.

2 Use each term-to-term rule and the given starting point to make a sequence with four terms in it.

a Rule: Subtract 3 Start at 21.
b Rule: Subtract 5 Start at 31.
c Rule: Divide by 5 Start at 250.
d Rule: Divide by 2 Start at 32.
e Rule: Subtract 8 Start at 36.
f Rule: Divide by 4 Start at 64.
g Rule: Divide by 2 Start at 8.
h Rule: Subtract 9 Start at 45.

3 Describe the term-to-term rule of each sequence below.
Use this rule to write down the next two terms in each sequence.

a 6, 8, 10, …, …
b 5, 10, 15, …, …
c 2, 20, 200, …, …
d 1, 4, 16, …, …
e 2, 9, 16, …, …
f 30, 23, 16, …, …
g 5, 10, 20, …, …
h 11, 33, 55, …, …

PS **4** Each of these sequences uses an ‘add’ rule.
Write down the rule for each one.
Copy and complete each sequence.

a 2, 5, 8, …, …, 17
b 1, 6, …, 16, …, …
c 5, …, …, 11, …, …
d …, 14, …, …, 29, …

PS **5** Each of these sequences uses a ‘multiply by’ rule.
Write down the rule for each one.
Copy and complete each sequence.

a 1, 10, …, …, 10 000
b 3, …, …, 24, 48, …
c …, …, …, 16, 32, …
d 4, …, …, 108, 324

PS **6** Each of these sequences uses a 'subtract' rule.
Write down the rule for each one.
Copy and complete each sequence.

a 52, 45, 38, …, …, 17 **b** 31, 26, …, 16, …, …
c 45, …, …, 15, … **d** …, 44, …, …, 26, …

PS **7** Each of these sequences uses a 'divide by' rule.
Write down the rule for each one.
Copy and complete each sequence.

a 80, 40, …, …, 5 **b** 81, …, …, 3, 1
c …, …, 8, …, …, 1 **d** 1250, …, …, 10, …

8 Write down the first five terms of each sequence, based on the rule.
Start each sequence with 2.

a Add 2 **b** Multiply by 4 **c** Add 5 **d** Multiply by 10
e Add 8 **f** Multiply by 5 **g** Add 3 **h** Multiply by 2

9 **a** The sequence below is part of a sequence with the rule 'Multiply by 2'.
…, 20, 40, 80, …
What are the two missing numbers?
b The sequence below is part of a sequence with the rule 'Divide by 2'.
…, 48, 24, 12, …, …
What are the three missing numbers?

Challenge: Rules for sequences

A For each pair of numbers below, write down two different rules to get from the first term to the second term. Then use each rule to write down the next two terms in the sequence.

a 1, 3, …, … **b** 2, 8, …, … **c** 5, 10, …, …
d 1, 5, …, … **e** 10, 20, …, … **f** 3, 12, …, …

B For each sequence below, write down a rule to get from the first term to the missing second term. Then use each rule to write down the first four terms of the sequence.

a 1, …, 7, … **b** 1, …, 11, … **c** 15, …, 25, …
d 6, …, 12, … **e** 5, …, 25, … **f** 18, …, 22, …

2.3 Finding terms in patterns

Learning objective

- To find missing terms in a sequence

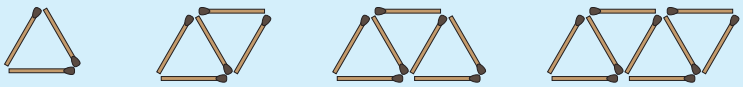
Key word

- differences

In any sequence, you will have a first term, second term, third term, fourth term and so on.

Example 4

Look at the pattern of matches, then draw the next pattern and state how many matches it will use.



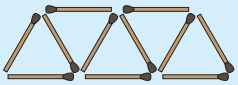
Pattern 1
3 matches

Pattern 2
5 matches

Pattern 3
7 matches

Pattern 4
9 matches

You can easily draw the next pattern and count that it has 11 matches.



Pattern 5
11 matches

You can carry on doing this for as many patterns as you like.

A better way to solve the last example is to put the numbers into a table, spot the term-to-term rule and use this to work out the number of matches.

Pattern	Matches
1	3
2	5
3	7
4	9
5	11

You can carry on with this table as far as you like, but there is an easier way to work out the number of matches in, for example, the 10th pattern.

The term-to-term rule is 'Add 2'. To get from the fifth pattern to the 10th pattern, you need to add on two five more times.

$$11 + 2 \times 5 = 11 + 10 = 21$$

So there are 21 matches in the 10th pattern.

However, it is even easier to start from the first term. To get to the 10th term you need to add on two nine (10 – 1) times.

$$3 + 2 \times (10 - 1) = 3 + 2 \times 9 = 3 + 18 = 21$$

To get to the 25th term you need to add on two twenty-four times.

$$3 + 24 \times 2 = 3 + 48 = 51$$

Exercise 2C

For each series of diagrams below:

a draw the next diagram

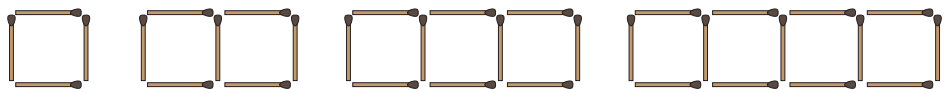
b write down the sequence to show the numbers of matches in the first five patterns

c work out how many matches there are in:

i the 10th

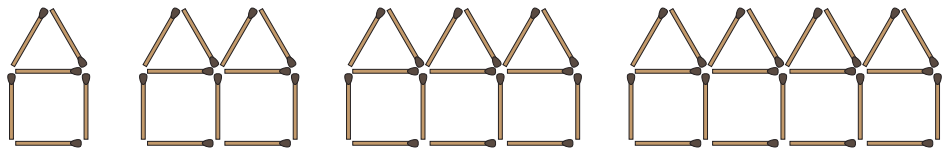
ii the 20th pattern.

1



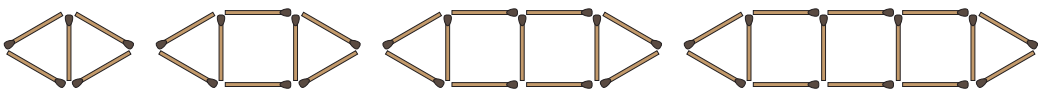
Pattern 1 Pattern 2 Pattern 3 Pattern 4

2



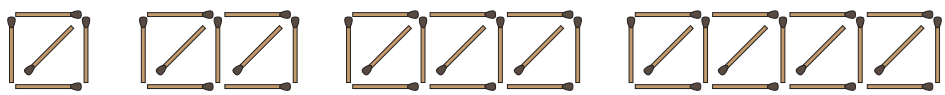
Pattern 1 Pattern 2 Pattern 3 Pattern 4

3



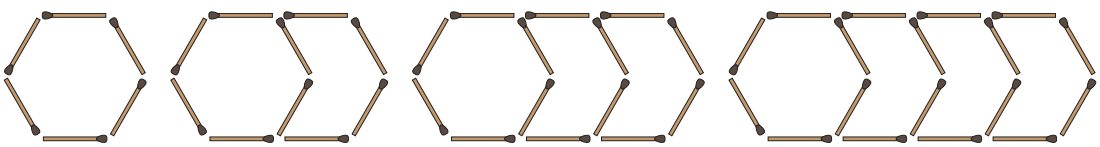
Pattern 1 Pattern 2 Pattern 3 Pattern 4

4



Pattern 1 Pattern 2 Pattern 3 Pattern 4

5



Pattern 1 Pattern 2 Pattern 3 Pattern 4

MR **6** For each sequence:

i write down the term-to-term rule

ii work out the 10th term.

a 2, 4, 6, 8, . . .

b 3, 8, 13, 18, . . .

c 4, 10, 16, 22, . . .

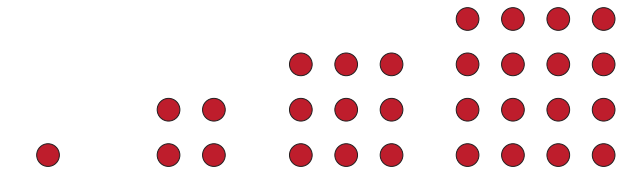
d 4, 8, 12, 16, . . .

e 7, 10, 13, 16, . . .

f 1, 9, 17, 25, . . .

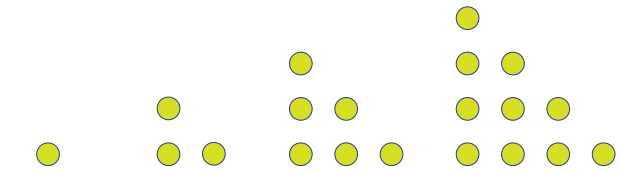
Challenge: Dot patterns

A How many dots will there be in: **a** the fifth pattern **b** the 10th pattern?



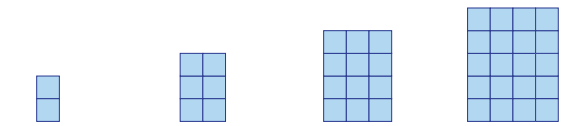
Pattern 1 Pattern 2 Pattern 3 Pattern 4

B How many dots will there be in: **a** the fifth pattern **b** the 10th pattern?



Pattern 1 Pattern 2 Pattern 3 Pattern 4

C How many squares will there be in: **a** the fifth pattern **b** the 10th pattern?



Pattern 1 Pattern 2 Pattern 3 Pattern 4

2.4 The square numbers

Learning objective

To introduce the sequence of square numbers

Key words	
square numbers	squaring

When you multiply a number by itself this is called **squaring** the number. The result is a **square number**. For example:

- 4 is a square number and is the result of squaring 2 ($2 \times 2 = 4$), so 4 is the square of 2
- 9 is a square number and is the result of squaring 3 ($3 \times 3 = 9$), so 9 is the square of 3

and so on.

Instead of writing 1×1, 2×2, $3 \times 3, \ldots$, you can write 1^2, 2^2, $3^2, \ldots$.

You read this as 'one squared, two squared, three squared…'.

This table shows the first seven square numbers.

1×1	2×2	3×3	4×4	5×5	6×6	7×7
1^2	2^2	3^2	4^2	5^2	6^2	7^2
1	4	9	16	25	36	49
						

You can see from the bottom line of the table why they are called square numbers.

This is the start of the sequence of square numbers.

You need to learn the square numbers up to $15^2 = 225$.

Exercise 2D

1 Continue the first three rows of the table of square numbers, above, up to 10×10.

2 Use a calculator to work out the rest of the square numbers up to 15×15.

3 Use a calculator work out these squares.

a 18^2 **b** 21^2 **c** 25^2

d 40^2 **e** 35^2 **f** 42^2

4 Write each number below as the addition of two square numbers.

The first two have been done for you.

a $5 = 1 + 4$ **b** $10 = 1 + 9$ **c** $13 = \ldots + \ldots$

d $17 = \ldots + \ldots$ **e** $20 = \ldots + \ldots$ **f** $25 = \ldots + \ldots$

PS **5** Look at this pattern of numbers.

$1 \quad = 1 = 1^2$

$1 + 3 \quad = 4 = 2^2$

$1 + 3 + 5 = 9 = 3^2$

a Write down the next two lines of this number pattern.

b What is special about the numbers in the middle column, between the equals signs?

c Without working them out, write down the answers to these calculations.

i $1 + 3 + 5 + 7 + 9 + 11 = \ldots$

ii $1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 = \ldots$

Challenge: Triples of squares

Some pairs of square numbers are special because when you add them they give an answer that is also a square number.

For example: $3^2 + 4^2 = 9 + 16 = 25 = 5^2$

Check that each of the following additions of square numbers also gives a square number.

In each case, write out the full calculation as above.

A $6^2 + 8^2$ **B** $5^2 + 12^2$

C $7^2 + 24^2$ **D** $10^2 + 24^2$

E $8^2 + 15^2$ **F** $9^2 + 12^2$

Hint

Find the button on your calculator that tests if a number is a square number.
It looks something like this.

If the answer is a decimal (not a whole number), then the number you put in is not a square number.

2.5 The triangular numbers

Learning objective

- To introduce the sequence of triangular numbers

Key words

triangular number

Another well-know sequence is:

1, 3, 6, 10, 15, 21, ...

This is called the sequence of **triangular numbers**.

This sequence builds up like this.

First term:		1
Second term:	Add 2	2 + 1 = 3
Third term:	Add 3 to the second term	3 + 3 = 6
Fourth term:	Add 4 to the third term	4 + 6 = 10
Fifth term:	Add 5 to the fourth term	5 + 10 = 15

This table shows you the first seven triangular numbers.

0 + 1	1 + 2	3 + 3	6 + 4	10 + 5	15 + 6	21 + 7
1	3	6	10	15	21	28
						

You can see from the bottom line of the table why they are called triangular numbers.

Exercise 2E

1 Continue the first two rows of the table of triangular numbers, above, up to the 10th triangular number.

2 Continue the sequence of triangular numbers up to the 15th triangular number.

3 Write each of these numbers as the addition of two triangular numbers. The first two have been done for you.

a $4 = 1 + 3$ **b** $7 = 1 + 6$ **c** $9 = \ldots + \ldots$

d $11 = \ldots + \ldots$ **e** $13 = \ldots + \ldots$ **f** $16 = \ldots + \ldots$

4 Write down two numbers that are both square numbers and also triangular numbers.

PS **5** Look at this pattern of numbers.

$1 = 1$

$1 + 2 = 3$

$1 + 2 + 3 = 6$

a Write down the next two lines of this number pattern.

b What is special about the numbers on the left-hand side?

c What is special about the numbers on the right-hand side?

d Without working them out, write down the answers to these calculations.

i $1 + 2 + 3 + 4 + 5 + 6 = \ldots$

ii $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 = \ldots$

6 **a** Add up the first five pairs of consecutive triangular numbers, starting with $1 + 3$, $3 + 6$, $6 + 10$, ...

b What is special about the answers?

Challenge: Triangular numbers

Here is the sequence of triangular numbers.

1, 3, 6 10, 15, 21, 28, ...

The first term is 1. You can work this out from: $0.5 \times 1 \times 2 = 1$

The second term is 3. You can work this out from: $0.5 \times 2 \times 3 = 3$

The third term is 6. You can work this out from: $0.5 \times 3 \times 4 = 6$

The fourth term is 10. You can work this out from: $0.5 \times 4 \times 5 = 10$

A Write down the next three lines in the sequence.

B Explain how you can use this pattern to work out a rule for any term in the sequence.

C Use your rule to work out:

a the 10th triangular number **b** the 20th triangular number

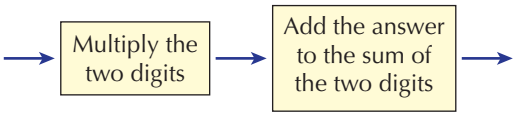
c the 50th triangular number **d** the 100th triangular number.

Investigation: A function machine problem

This is an investigation of a double function machine that uses a two-digit whole-number input.

Examples of two-digit whole numbers are 12, 36, 45, 71, 98.

The double function machine is:



For example: start with an input of 23.

$23 \rightarrow 2 \times 3 = 6 \rightarrow 6 + 2 + 3 = 11$ So the output is 11.

Now start the investigation.

If a two-digit number input gives an output that is a single-digit number, then stop.

For example, start with an input of 14.

$14 \rightarrow 1 \times 4 = 4 \rightarrow 4 + 1 + 4 = 9$, so stop as the output is a single-digit number.

If a two-digit number gives an output that is a two-digit number, then repeat until you reach a single-digit number.

For example, start with an input of 83.

$83 \rightarrow 8 \times 3 = 24 \rightarrow 24 + 8 + 3 = 35$ Now repeat with an input of 35.

$35 \rightarrow 3 \times 5 = 15 \rightarrow 15 + 3 + 5 = 23$ Now repeat with an input of 23.

$23 \rightarrow 2 \times 3 = 6 \rightarrow 6 + 2 + 3 = 11$ Now repeat with an input of 11.

$11 \rightarrow 1 \times 1 = 1 \rightarrow 1 + 1 + 1 = 3$ Stop, as the output is a single-digit number.

A Start with an input of 31. Work out the output.

B Start with an input of 24. Work out the output.

C Start with an input of 66. Work out the output.

D Start with an input of 19. What happens?

E Start with another input that ends with 9. What happens?

F Start with an input of 91. What happens?

G Start with another input that starts with 9. What happens?

H **a** Start with an input of 56. What is the output?

b Start with an input of 65. Explain how you know what the output is, without working it out.

Ready to progress?

I can find the output for a single function machine when I know the input value.

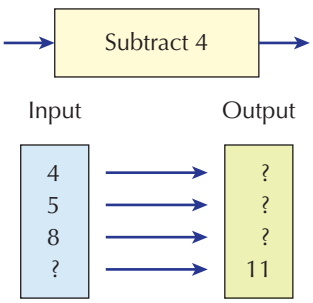
I can find the output for a double function machine when I know the input value.
I can write down a sequence, given the first term and a term-to-term rule.
I can give the term-to-term rule for a sequence.
I know how to work out square numbers and triangular numbers.

I can find any term in a sequence, given the first term and a term-to-term rule.

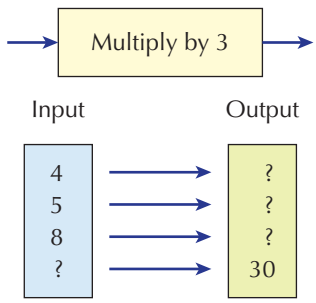
Review questions

1 Work out the missing inputs and outputs for each function machine.

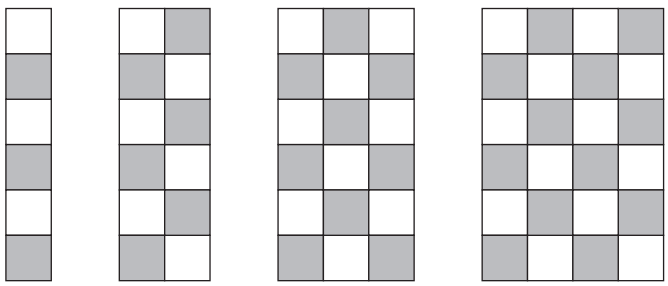
a



b



2 Here is a sequence of shapes made with grey and white tiles.



The number of grey tiles = 3 × the shape number

The number of white tiles = 3 × the shape number

a How many white tiles will there be in shape number 8?

b Altogether, how many tiles will there be in shape number 8?

c Altogether, how many tiles will there be in shape number 15?

d Write down the missing number from this sentence:
The total number of tiles = . . . × the shape number.

3 Each of these double function machines can be replaced with a single function machine.

Work out the operation that will go in the single function machine. The first one has been done for you.

a → Add 3 → Add 6 → = → Add 9 →

b → Add 6 → Subtract 3 → = → ? →

c → Multiply by 2 → Multiply by 4 → = → ? →

d → Multiply by 6 → Divide by 2 → = → ? →

4 **a** Use the term-to-term rule to write the next two numbers in each sequence.

i Rule: → Add 6 → 4 10 … …

ii Rule: → Multiply by 4 → 1 4 … …

iii Rule: → Multiply by 2 → Add 3 → 4 11 … …

b A sequence of numbers starts like this:

30, 22, 14

The rule is 'Subtract 8'.

Work out the first negative number in the sequence.

FS

5 **a** Jeni saves £20 each week for 8 weeks. How much will she have saved after:
i 2 weeks **ii** 8 weeks?

b Jeni's friend Lucie already has £16 saved. She then saves £9 the first week, £11 the second week, £13 the third week and so on for 8 weeks.

Copy and complete this table for Lucie.

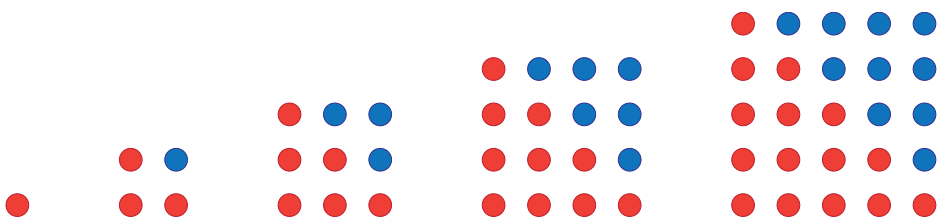
Week	1	2	3	4	5	6	7	8
Amount saved (£)	9	11	13	15	17			
Total amount saved (£)	25	36						

c **i** Who will have more money at the end of eight weeks?

ii How much more will she have than her friend?

MR

6 This pattern of square numbers is made with coloured counters.



a Write down the sequence formed by the red counters.

b Write down the sequence formed by the blue counters.

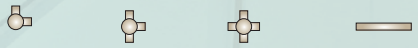
c Write 49 as the sum of two triangular numbers.

d Write 100 as the sum of two triangular numbers.

Mathematical reasoning

Valencia Planetarium

Grids are made from combinations of these rods and links.



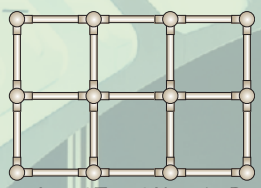
L links T links X links R rods

Examples

This grid is made from 4 L links + 2 T links + 7 R rods.

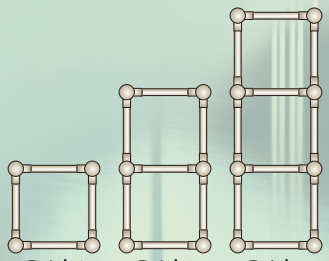
$4L + 2T + 7R$

This grid is made from 4 L links + 6 T links + 2 X links + 17 R rods.



$4L + 6T + 2X + 17R$

1 Look at the grids on the right.



Grid 1 Grid 2 Grid 3

a Copy and complete this table.

Grid	L links	T links	R rods
1	4	0	4
2	4	2	7
3			

b Continue the table, to find the numbers of links and rods up to grid 6.

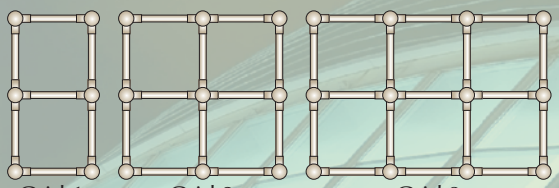
c Copy and complete these statements.

L links = …

T links = grid number × 2 – …

R rods = grid number × … + …

2 Look at these grids.



a Copy and complete this table.

Grid	L links	T links	X links	R rods
1	4	2	0	7
2	4	4	1	12
3				
4				

b Copy and complete these statements.

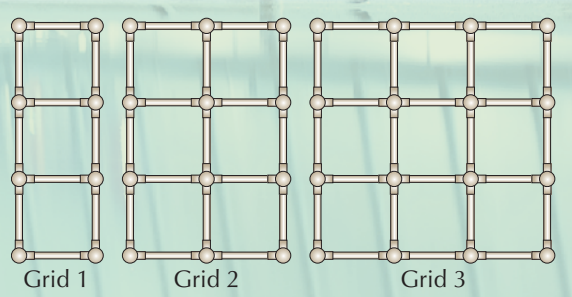
L links = …

T links = grid number × …

X links = grid number × 1 – …

R rods = grid number × … + …

3 Look at these grids.



a Copy and complete this table.

Grid	L links	T links	X links	R rods
1	4	4	0	10
2	4	6	2	17
3				
4				

b Copy and complete these statements.

L links = …

T links = grid number × … + …

X links = grid number × 2 – …

R rods = grid number × … + …

c How many links and rods will there be in grid 10?