Worked solutions

1 Algebra and functions 1: Manipulating algebraic expressions

Prior knowledge 1 page 2

1 a $3(a+b)=3a+3b$

b $c(4-d)=4c-cd$

c $e^2(f-g+eh)=e^2f-e^2g+e^3h$

d $i(5+j)+7(k-2j)=5i+ij+7k-14j$

e $l(l+5)-6(l+l^2)=l^2+5l-6l-6l^2=-5l^2-l$

f $m(2m-n)-2(m+n)=2m^2-mn-2m-2n$

2 a $(x+1)(x+2)=x^2+3x+2$

b $(x-3)(x+4)=x^2+x-12$

c $(x-2)(x-3)=x^2-5x+6$

d $(x-3)^2=x^2-6x+9$

e $(2x+1)(x+2)=2x^2+5x+2$

f $(x+1)(x+2)(x+3)=x^3+6x^2+11x+6$

3 a $a^5\times a^6=a^{11}$

b $a^{\frac{1}{2}}\times a^{\frac{1}{2}}=a$

c $\frac{b^9}{b^5}=b^4$

d $(c^{-3})^2=c^{-6}$

e $(2d)^4=16d^4$

f $\frac{(e^2\times f^8)}{ef}=ef^7$

4 a $\sqrt{2}\times\sqrt{3}=\sqrt{6}$

b $\sqrt{5}\times\sqrt{8}\times\sqrt{3}=2\sqrt{30}$

c $\sqrt{72}=6\sqrt{2}$

d $\frac{\sqrt{54}}{\sqrt{3}}=3\sqrt{2}$

e $\frac{1}{\sqrt{5}}=\frac{\sqrt{5}}{5}$

f $\frac{\sqrt{8}\times\sqrt{6}}{\sqrt{12}}=2$

Exercise 1.1A page 5

1 a $4x^3+12x^2-8x$

b $x^3+x^2y-x^3y$

2 a x^3+x^2y-x-y

b $6x-x^2-x^3$

3 a x^3+3x^2-4

b $6+4x-6x^2-4x^3$

4 $(2x-1)(x+2)-(1-x)(3x^2+6)=2x^2+3x-2$
$-(3x^2+6-3x^3\ 6x)$
$=2x^2+3x-2$
$-3x^2-6+3x^3+6x$
$=3x^3-x^2+9x-8$

5 $(2n)^2+(2n+2)^2=4n^2+4n^2+8n+4$
$=8n^2+8n+4$
$=4(2n^2+2n+1)$

Since 4 is a multiple of 2, $4(2n^2+2n+1)$must be an even number.

6 $(x+3)(2x^2+x-2)-(x+1)(3-2x-x^2)$
$=x(2x^2+x-2)+3(2x^2+x-2)$
$-x(3-2x-x^2)+1(3-2x-x^2)$

Should be:

$=x(2x^2+x-2)+3(2x^2+x-2)$
$-x(3-2x-x^2)-1(3-2x-x^2)$
$=2x^2+x^2-2x+3x^2+3x-6-3x+2x^2+x^3+3-2x-x^2$

Should be:

$=2x^3+x^2-2x+6x^2+3x-6-3x+2x^2+x^3-3+2x+x^2$
$=3x^2-6x+3$

Should be:

$=3x^3+10x^2-9$

7 Let $p=2n+1$ where n is an integer. Consequently p is an odd number.

$p^2=(2n+1)^2$
$=4n^2+4n+1$
$=4(n^2+n)+1$

Since $4(n^2+2n)$ is even, $4(n^2+n)+1$ is odd.

8 $2p(2n+1)(2m+1)=2p(4mn+2n+2m+2)$
$=8mnp+4pn+4mp+4p$
$=2(4mnp+2pn+2mp+2p)$, an even number

Exercise 1.2A page 8

1 a $(x+2)^3=(1)(x)^3(2)^0+(3)(x)^2(2)^1+(3)(x)^1(2)^2+(1)(x)^0(2)^3$
$=x^3+6x^2+6x+8$

b $(2x+1)^4=(1)(2x)^4(1)^0+(4)(2x)^3(1)^1+(6)(2x)^2(1)^2+(4)(2x)^1(1)^3+(1)(2x)^0(1)^4$
$=16x^4+32x^3+24x^2+8x+1$

2 **a** $(2-x)^5 = (1)(2)^5(-x)^0 + (5)(2)^4(-x)^1 + (10)(2)^3(-x)^2 + (10)(2)^2(-x)^3 + (5)(2)^1(-x)^4 + (1)(2)^0(-x)^5$

$= 32 - 80x + 80x^2 - 40x^3 + 10x^4 - x^5$

b $\left(\frac{x}{2}+3\right)^4 = (1)\left(\frac{x}{2}\right)^4(3)^0 + (4)\left(\frac{x}{2}\right)^3(3)^1 + (6)\left(\frac{x}{2}\right)^2(3)^2 + (4)\left(\frac{x}{2}\right)^1(3)^3 + (1)\left(\frac{x}{2}\right)^0(3)^4$

$= \frac{x^4}{16} + \frac{3x^3}{2} + \frac{27x^2}{2} + 54x + 81$

3 $(x-1)^4 = (1)(x)^4(-1)^0 + (4)(x)^3(-1)^1 + (6)(x)^2(-1)^2 + (4)(x)^1(-1)^3 + (1)(x)^0(-1)^4$

$= x^4 - 4x^3 + 6x^2 - 4x + 1$

$(3x+2)(x-1)^4 = (3x+2)(x^4 - 4x^3 + 6x^2 - 4x + 1)$

$= 3x^5 - 10x^4 + 10x^3 - 5x + 2$

4 $(10)(3)^3(-2x)^2 = 1080x^2$

5 $(x^3 - 2x^2 + \frac{1}{2})^3 = (1)(x^3-2x^2)^3(\frac{1}{2})^0 + (3)(x^3-2x^2)^2(\frac{1}{2})^1 + (3)(x^3-2x^2)^1(\frac{1}{2})^2 + (1)(x^3-2x^2)^0(\frac{1}{2})^3$

$(x^3-2x^2)^3 = (1)(x^3)^3(-2x^2)^0 + (3)(x^3)^2(-2x^2)^1 + (3)(x^3)^1(-2x^2)^2 + (1)(x^3)^0(-2x^2)^3$

$= x^9 - 6x^8 + 12x^7 - 8x^6$

$(x^3 - 2x^2 + \frac{1}{2})^3 = x^9 - 6x^8 + 12x^7 - \frac{13}{2}x^6 - 6x^5 + 6x^4 + \frac{3x^3}{4} - \frac{3x^2}{2} + \frac{1}{8}$

6 $198x^2 = (2x)(3)(2)^2(cx)^1 + (5)(3)(2)^1(cx)^2$

$198x^2 = 24cx^2 + 30c^2x^2$

$198 = 24c + 30c^2$

$5c^2 + 4c - 33 = 0$

$(5c-11)(c+3) = 0$

$c = \frac{11}{5}$ or $c = -3$

Exercise 1.2B — page 11

1 **a** 10 **b** 15 **c** 8

2 **a** 1 **b** 3 **c** 3 **d** 1

They are the coefficients for a cubic expansion from Pascal's triangle.

3 ${}^5C_0, {}^5C_1, {}^5C_2, {}^5C_3, {}^5C_4, {}^5C_5$

4 All the rows prior to the index required have to be written in Pascal's triangle: a laborious and error-prone exercise.

5 ${}^{16}C_9 = \frac{16!}{7!9!}$

$= \frac{16\times15\times14\times13\times12\times11\times10}{7\times6\times5\times4\times3\times2\times1}$

$= 4\times1\times2\times13\times2\times11\times5$

$= 11\,440$

Exercise 1.3A — page 14

1 **a** $(2+x)^4 = {}^4C_0 2^4 + {}^4C_1 2^3(x)^1 + {}^4C_2 2^2(x)^2 + {}^4C_3 2^1(x)^3 + {}^4C_4(x)^4$

$= \frac{4!}{4!0!}(16) + \frac{4!}{3!1!}(8)(x) + \frac{4!}{2!2!}(4)(x^2) + \frac{4!}{1!3!}(2)(x^3) + \frac{4!}{0!4!}(x^4)$

$= 16 + 32x + 24x^2 + 8x^3 + x^4$

b $(3+5x)^3 = {}^3C_0 3^3 + {}^3C_1 3^2(5x)^1 + {}^3C_2 3^1(5x)^2 + {}^3C_3(5x)^3$

$= \frac{3!}{3!0!}(27) + \frac{3!}{2!1!}(9)(5x) + \frac{3!}{1!2!}(3)(25x^2) + \frac{3!}{0!3!}(125x^3)$

$= 27 + 135x + 225x^2 + 125x^3$

2 **a** $(1+3x)^5 = \frac{5(5-1)(5-2)}{3!}(3x)^3$

$= 270x^3$

b $(2+4x)^6 = 2^6(1+2x)^6$

$= 2^6\frac{6(6-1)(6-2)}{3!}(2x)^3$

$= 10\,240x^3$

3 **a** $\left(2+\frac{x}{2}\right)^3 = {}^3C_0 2^3 + {}^3C_1 2^2\left(\frac{x}{2}\right) + {}^3C_2 2^1\left(\frac{x}{2}\right)^2 + {}^3C_3\left(\frac{x}{2}\right)^3$

$= \frac{3!}{3!0!}(8) + \frac{3!}{2!1!}(4)\left(\frac{x}{2}\right) + \frac{3!}{1!2!}(2)\left(\frac{x^2}{4}\right) + \frac{3!}{0!3!}\left(\frac{x^3}{8}\right)$

$= 8 + 6x + \frac{3x^2}{2} + \frac{x^3}{8}$

b $(3x-6y)^4 = {}^4C_0(3x)^4 + {}^4C_1(3x)^3(-6y)^1 + {}^4C_2(3x)^2(-6y)^2 + {}^4C_3(3x)^1(-6y)^3 + {}^4C_4(-6y)^4$

$= \frac{4!}{4!0!}(81x^4) + \frac{4!}{3!1!}(27x^3)(-6y) + \frac{4!}{2!2!}(9x^2)(36y^2) + \frac{4!}{1!3!}(3x)(-216y^3) + \frac{4!}{0!4!}(1296y^4)$

$= 81x^4 - 648x^3y + 1944x^2y^2 - 2592xy^3 + 1296y^4$

4 **a** $(3+x)^5 = 3^5(1+\frac{x}{3})^5$

$= 3^5\frac{5(5-1)(5-2)}{3!}\left(\frac{x}{3}\right)^3$

$= 90x^3$

b $(2+3x)^6 = 2^6\left(1+\frac{3x}{2}\right)^6$

$= 2^6\frac{6(6-1)(6-2)}{3!}\left(\frac{3x}{2}\right)^3$

$= 4320x^3$

5 $(2+bx)^7 = 2^7\left(1+\frac{bx}{2}\right)^7$

$= 2^7\frac{7(7-1)(7-2)}{3!}\left(\frac{bx}{2}\right)^3$

$560b^3x^3 = 70\,000x^3$

$b^3 = 125$

$b = 5$

6 $\left(3-\frac{2x}{5}\right)^4 = {}^4C_0(3)^4 + {}^4C_1(3)^3\left(-\frac{2x}{5}\right)^1 + {}^4C_2(3)^2\left(-\frac{2x}{5}\right)^2$

$+{}^4C_3(3)^1\left(-\frac{2x}{5}\right)^3 + {}^4C_4\left(-\frac{2x}{5}\right)^4$

$= \frac{4!}{4!0!}(81) + \frac{4!}{3!1!}(27)\left(-\frac{2x}{5}\right) + \frac{4!}{2!2!}(9)\left(\frac{4x^2}{25}\right)$

$+\frac{4!}{1!3!}(3)\left(-\frac{8x^3}{125}\right) + \frac{4!}{0!4!}\left(\frac{16x^4}{625}\right)$

$= 81 - \frac{216x}{5} + \frac{216x^2}{25} - \frac{96x^3}{125} + \frac{16x^4}{625}$

$(3-x)\left(3-\frac{2x}{5}\right)^4 = (3-x)\left(81 - \frac{216x}{5} + \frac{216x^2}{25} - \frac{96x^3}{125} + \frac{16x^4}{625}\right)$

$= 243 - \frac{648x}{5} + \frac{648x^2}{25} - \frac{288x^3}{125} + \frac{48x^4}{625}$

$-81x + \frac{216x^2}{5} - \frac{216x^3}{25} + \frac{96x^4}{125} - \frac{16x^5}{625}$

$= 243 - \frac{1053x}{5} + \frac{1728x^2}{25} - \frac{1368x^3}{125}$

$+\frac{528x^4}{625} - \frac{16x^5}{625}$

7 $(2x)^4C_2(7)^2\left(-\frac{x}{2}\right)^2 + (5)^4C_3(7)^1\left(-\frac{5x}{2}\right)^3$

$= (2x)\frac{4!}{2!2!}(49)\left(\frac{x^2}{4}\right) + (5)\frac{4!}{1!3!}(7)\left(-\frac{x^3}{8}\right)$

$= 147x^3 - \frac{35x^3}{2}$

$= \frac{259x^3}{2}$

8 $\left(3-\frac{5x}{2}\right)^6 = {}^6C_0(3)^6 + {}^6C_1(3)^5\left(-\frac{5x}{2}\right)^1$

$+{}^6C_2(3)^4\left(-\frac{5x}{2}\right)^2 + {}^6C_3(3)^3\left(-\frac{5x}{2}\right)^3$

$= \frac{6!}{6!0!}(729) + \frac{6!}{5!1!}(243)\left(-\frac{5x}{2}\right)$

$+\frac{6!}{4!2!}(81)\left(\frac{25x^2}{4}\right) + \frac{6!}{3!3!}(27)\left(-\frac{125x^3}{8}\right)$

$= 729 - 3645x + \frac{30375x^2}{4} - \frac{16875x^3}{2}$

$x = 0.1$

$2.75^6 \approx 729 - 364.5 + 75.9375 - 8.4375$

≈ 432

Exercise 1.4A page 16

1 **a** $2(x+3)$ **b** $3(x^3+3)$

c $4x(2-x^3)$

d $xy + 2xy - xz + 3yz$ doesn't factorise.

2 **a** $(x-3)(x-4)$ **b** $(2x+1)(x+2)$

c $(x+4)(x-3)$ **d** $(x-7)(x+7)$

3 $14x^2 - 56x = 7x(2x^2 - 8)$

Should be:

$= 7x(2x-8)$

4 **a** $4xy(1-2y)$ **b** $(2x-3)(3x-2)$ **c** $xy^2(x+2y)$

d $6x(3x-2)$ **e** $(9xy+7z)(9xy-7z)$

f $x(2x^2-7x+10)$ the quadratic cannot be factorised.

5 $4x^2 + 6x - 4 = (4x-2)(x+2)$ or $(2x-1)(2x+4)$

6 $(7x+2y)(2x-3y)$

7 $(6x+1)(x+15)$ $(6x+15)(x+1)$

$(6x+3)(x+5)$ $(6x+5)(x+3)$

$(2x+1)(3x+15)$ $(2x+15)(3x+1)$

$(2x+3)(3x+5)$ $(2x+5)(3x+3)$

$(3x+1)(2x+15)$ $(3x+15)(2x+1)$

$(3x+3)(2x+5)$ $(3x+5)(2x+3)$

8 $p(q-r^2) = 2r(3p-q)$

$pq - pr^2 = 6pr - 2qr$

$p(q-r^2-6r) = -2rq$

$p = \frac{-2qr}{q-r^2-6r}$ or $p = \frac{2qr}{r^2+6r-q}$

9 $(x-3)^2 + (y+5)^2 = 16$

Exercise 1.5A page 20

1 **a** 2 **b** $\frac{x+2}{x+3}$

2 $f(1) = 1^3 + 4(1^2) + 1 - 6 = 0$ so $(x-1)$ is a factor of $f(x)$.

$x^3 + 4x^2 + x - 6 = (x-1)(x+2)(x+3)$

3 **a** $\frac{x^2+x-6}{x^2+8x+15} = \frac{(x+3)(x-2)}{(x+3)(x+5)} = \frac{x+2}{x+5}$

b $\frac{2x^2-5x+2}{2x^2+5x-3} = \frac{(2x-1)(x-2)}{(2x-1)(x+3)} = \frac{x-2}{x+3}$

4 **a** $f(x) = 3x^3 + 2x^2 - 7x + 2$

$f(1) = 3 + 2 - 7 + 2 = 0$ so $(x-1)$ is a factor.

$3x^3 + 2x^2 - 7x + 2 = (x-1)(3x-1)(x+2)$

b $f(x) = 2x^4 + 3x^3 - 12x^2 - 7x + 6$

$f(-1) = 2 - 3 - 12 + 7 + 6 = 0$ so $(x + 1)$ is a factor.

$2x^4 + 3x^3 - 12x^2 - 7x + 6 = (x + 1)(2x^3 + x^2 - 13x + 6)$

$g(x) = 2x^3 + x^2 - 13x + 6$

$g(-3) = -54 + 9 + 39 + 6 = 0$ so $(x + 3)$ is a factor.

$2x^4 + 3x^3 - 12x^2 - 7x + 6 = (x + 1)(x + 3)(2x - 1)(x - 2)$

c $f(x) = x^4 - 25x^2 + 144$

$f(4) = 256 - 400 + 144 = 0$ so $(x - 4)$ is a factor.

$x^4 - 25x^2 + 144 = (x - 4)(x^3 + 4x^2 - 9x - 36)$

$g(x) = x^3 + 4x^2 - 9x - 36$

$g(-3) = (-3)^3 + 4(-3)^2 - 9(-3) - 36 = 0$ so $(x + 3)$ is a factor.

$x^4 - 25x^2 + 144 = (x - 4)(x + 4)(x + 3)(x - 3)$

5 **a** $2x^3 + 15x^2 + 31x + 12 = (Ax^2 + Bx + C)(x - 1) + D$

$x = 1: 2 + 15 + 31 + 12 = D$

$D = 60$

$x = 0: 12 = (C)(-1) + 60$

$C = 48$

Equating coefficients:

$x^3: 2 = A$

$x^2: 15 = -A + B$

$B = 17$

$2x^3 + 15x^2 + 31x + 12 = (2x^2 + 17x + 48)(x - 1) + 60$

b $x^3 + 6x^2 + 11x + 6 = (Ax^2 + Bx + C)(x - 2) + D$

$x = 2: 8 + 24 + 22 + 6 = D$

$D = 60$

$x = 0: 6 = (C)(-2) + 60$

$C = 27$

Equating coefficients:

$x^3: 1 = A$

$x^2: 6 = -2A + B$

$B = 8$

$x^3 + 6x^2 + 11x + 6 = (x^2 + 8x + 27)(x - 2) + 60$

c $x^3 + 4x^2 + x - 6 = (Ax^2 + Bx + C)(x + 4) + D$

$x = -4: -64 + 64 - 4 - 6 = D$

$D = -10$

$x = 0: -6 = (C)(4) - 10$

$C = 1$

Equating coefficients:

$x^3: 1 = A$

$x^2: 4 = 4A + B$

$B = 0$

$x^3 + 4x^2 + x - 6 = (x^2 + 1)(x + 4) - 10$

6 **a** $f(x) = x^4 - 18x + 81$

$f(2) = 16 - 36 + 81 = 61$

b $f(x) = 8x^4 + 2x^3 - 53x^2 + 37x - 6$

$f(-1) = 8 - 2 - 53 - 37 - 6 = -90$

c $f(x) = 2x^5 - 19x^4 + 60x^3 - 65x^2 - 2x + 24$

$f(-2) = -64 - 304 - 480 - 260 + 4 + 24 = -1080$

7 **a** $x^4 - 18x + 81 = (Ax^3 + Bx^2 + Cx + D)(x - 2) + E$

$x = 2: 16 - 36 + 81 = E$

$E = 61$

$x = 0: 81 = (D)(-2) + 61$

$D = -10$

Equating coefficients:

$x^4: 1 = A$

$x^3: 0 = -2A + B$

$B = 2$

$x^2: 0 = -2B + C$

$C = 4$

$x^4 - 18x + 81 = (x^3 + 2x^2 + 4x - 10)(x - 2) + 61$

b $8x^4 + 2x^3 - 53x^2 + 37x - 6$
$= (Ax^3 + Bx^2 + Cx + D)(x + 1) + E$

$x = -1: 8 - 2 - 53 - 37 - 6 = E$

$E = -90$

$x = 0: -6 = (D)(1) - 90$

$D = 84$

Equating coefficients:

$x^4: 8 = A$

$x^3: 2 = A + B$

$B = -6$

$x^2: -53 = B + C$

$C = -47$

$8x^4 + 2x^3 - 53x^2 + 37x - 6$
$= (8x^3 - 6x^2 - 47x + 84)(x + 1) - 90$

8 $f(x) = x^4 - 7x^3 + 13x^2 + 3x - 18$

$f(2) = 16 - 56 + 52 + 6 - 18 = 0$ so $(x - 2)$ is a factor.

$x^4 - 7x^3 + 13x^2 + 3x - 18 = (x - 2)(x^3 - 5x^2 + 3x + 9)$

$g(x) = x^3 - 5x^2 + 3x + 9$

$g(3) = 27 - 45 + 9 + 9 = 0$ so $(x - 3)$ is a factor.

$g(x) = (x - 3)(x^2 - 2x - 3)$

$f(x) = (x + 1)(x - 2)(x - 3)^2$

Exercise 1.6A page 23

1 **a** $2a^9$ **b** $2b^4$ **c** $3c^3$
d $27d^6$ **e** e^{-6} **f** f

2 **a** $64^{\frac{1}{3}}=\sqrt[3]{64}=4$ **b** $25^{\frac{1}{2}}=\sqrt{25}=\pm 5$

c $27^{-\frac{1}{3}}=\frac{1}{\sqrt[3]{27}}=\frac{1}{3}$

3 **a** $6a^4$ **b** $2b^5$ **c** $\frac{1}{2}c^{-4}$ or $\frac{1}{2c^4}$

d $d^{\frac{17}{12}}$ **e** $90e^3$ **f** $\frac{1}{3}f^{-2}g^{-2}$ or $\frac{1}{3f^2g^2}$

4 **a** $64^{-\frac{2}{3}}=\frac{1}{\sqrt[3]{64^2}}=\frac{1}{16}$ **b** $(-6)^{-2}=\frac{1}{(-6)^2}=\frac{1}{36}$

c $256^{\frac{1}{4}}=\sqrt[4]{256}=\pm 4$

5 $(3x^2)^3\times 2x^{-4}$

$=3x^6\times 2x^{-4}$

$=5x^{10}$

Should be:

$=27x^6\times 2x^{-4}$

$=54x^2$

6 $256=2^8$

$8=2^3$

$256=2^3\times 2^3\times 2^2$

$=8^1\times 8^1\times 8^{\frac{2}{3}}$

$=8^{\frac{8}{3}}$

7 Area $=\pi\left(2x^{-\frac{1}{3}}\right)^2$

$=4\pi x^{-\frac{2}{3}}$ or $\frac{4\pi}{x^{\frac{2}{3}}}$

8 **a** $\frac{9}{2}a^{-3}$ or $\frac{9}{2a^3}$ **b** $\frac{1}{216}b^{-\frac{1}{3}}$ or $\frac{1}{216b^{\frac{1}{3}}}$

c $16c^2\times 2c^{\frac{3}{2}}=32c^{\frac{7}{2}}$

9 Volume $=\frac{4}{3}\pi\left(\frac{3}{2}x^{\frac{1}{2}}\right)^3$

$=\frac{9}{2}\pi x^{\frac{3}{2}}$

10 $y=x^3$ and $z=x^5$

$\sqrt[3]{y}=x$

$\sqrt[5]{z}=x$

$y^{\frac{1}{3}}=z^{\frac{1}{5}}$

$z=y^{\frac{5}{3}}$

Exercise 1.7A page 25

1 **a** $3a^2$ **b** $5b$

2 **a** $3\sqrt{e}+e$ **b** $8\sqrt{f}-6f$

3 **a** $7c^2$ **b** $3d\sqrt{3d}$

4 **a** $g\sqrt{h}+h\sqrt{gh}$ **b** $j\sqrt{k}-k\sqrt{2j}$

5 $b-a^2$

6 $(\sqrt{b}-a)^2=(\sqrt{b}-a)(\sqrt{b}-a)$

$=b-a\sqrt{b}-a\sqrt{b}+a^2$

$=b-2a\sqrt{b}+a^2$

7 $(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b})=a-b$

8 $(\sqrt{ab}+\sqrt{a})^2=(\sqrt{ab}+\sqrt{a})(\sqrt{ab}+\sqrt{a})$

$=ab+a\sqrt{b}+a\sqrt{b}+a$

$=ab+2a\sqrt{b}+a$

9 $a-ab=(\sqrt{a}-\sqrt{ab})(\sqrt{a}+\sqrt{ab})$

10 $b^2-ab=(b-\sqrt{ab})(b+\sqrt{ab})$

Exercise 1.8A page 27

1 **a** $\frac{2}{\sqrt{3}}\times\frac{\sqrt{3}}{\sqrt{3}}=\frac{2\sqrt{3}}{3}$

b $\frac{4}{2+\sqrt{3}}\times\frac{2-\sqrt{3}}{2-\sqrt{3}}=8-4\sqrt{3}$

c $\frac{5x}{3-\sqrt{2}}\times\frac{3+\sqrt{2}}{3+\sqrt{2}}=\frac{15x+5x\sqrt{2}}{7}$

2 **a** $\frac{2\sqrt{2}}{3-\sqrt{2}}\times\frac{3+\sqrt{2}}{3+\sqrt{2}}=\frac{6\sqrt{2}+4}{7}$

b $\frac{4x+5}{3-\sqrt{7}}\times\frac{3+\sqrt{7}}{3+\sqrt{7}}=\frac{12x+4x\sqrt{7}+15+5\sqrt{7}}{2}$

c $\frac{x\sqrt{2}}{5+\sqrt{2}}\times\frac{5-\sqrt{2}}{5-\sqrt{2}}=\frac{5x\sqrt{2}-2x}{23}$

3 **a** $\frac{3-\sqrt{7}}{3+\sqrt{7}}\times\frac{3-\sqrt{7}}{3+\sqrt{7}}=\frac{16-6\sqrt{7}}{2}$

b $\frac{5+\sqrt{3}}{5-\sqrt{3}}\times\frac{5+\sqrt{3}}{5+\sqrt{3}}=\frac{28+10\sqrt{3}}{22}$

c $\frac{y-x}{y+\sqrt{x}}\times\frac{y-\sqrt{x}}{y-\sqrt{x}}=\frac{y^2-y\sqrt{x}-xy+x\sqrt{x}}{y^2-x}$

4 **a** $\frac{2}{\sqrt{2}}=\frac{2^1}{2^{\frac{1}{2}}}=2^{\frac{1}{2}}$ **b** $\frac{1}{2\sqrt{2}}=\frac{1}{2^1\times 2^{\frac{1}{2}}}=2^{-\frac{3}{2}}$

c $\frac{\sqrt[3]{64}}{2\sqrt{2}}=\frac{2^2}{2^1\times 2^{\frac{1}{2}}}=2^{\frac{1}{2}}$

5 **a** $\frac{x\sqrt{x}}{2-\sqrt{x}}\times\frac{2+\sqrt{x}}{2+\sqrt{x}}=\frac{2x\sqrt{x}+x^2}{4-x}$

b $\frac{y+\sqrt{x}}{y-\sqrt{x}}\times\frac{y+\sqrt{x}}{y+\sqrt{x}}=\frac{y^2+2y\sqrt{x}+x}{y^2-x}$

c $\frac{\sqrt{x}-2}{-3-\sqrt{y}}\times\frac{3-\sqrt{y}}{3-\sqrt{y}}=\frac{3\sqrt{x}-\sqrt{xy}-6+2\sqrt{y}}{y-9}$

6 a $\frac{x}{\sqrt{2}}+\frac{y}{3}=\left(\frac{x}{\sqrt{2}}\times\frac{\sqrt{2}}{\sqrt{2}}\right)+\frac{y}{3}$

$=\frac{x\sqrt{2}}{2}+\frac{y}{3}$

$=\frac{3x\sqrt{2}+2y}{6}$

b $\frac{\sqrt{x}}{1-\sqrt{x}}+\frac{\sqrt{y}}{2}=\left(\frac{\sqrt{x}}{1-\sqrt{x}}\times\frac{1+\sqrt{x}}{1+\sqrt{x}}\right)+\frac{\sqrt{y}}{2}$

$=\frac{\sqrt{x}+x}{1-x}+\frac{\sqrt{y}}{2}$

$=\frac{2\sqrt{x}+2x+\sqrt{y}-x\sqrt{y}}{2-2x}$

c $\frac{2-\sqrt{x}}{-\sqrt{x}-2}-\frac{\sqrt{x}+2}{\sqrt{x}}$

$=\left(\frac{2-\sqrt{x}}{-\sqrt{x}-2}\times\frac{\sqrt{x}-2}{\sqrt{x}-2}\right)-\left(\frac{\sqrt{x}+2}{\sqrt{x}}\times\frac{\sqrt{x}}{\sqrt{x}}\right)$

$=\frac{4\sqrt{x}-4-x}{4-x}-\frac{x+2\sqrt{x}}{x}$

$=\frac{4x\sqrt{x}-4x-x^2-4x+x^2-8\sqrt{x}+2x\sqrt{x}}{4x-x^2}$

$=\frac{6x\sqrt{x}-8x-8\sqrt{x}}{4x-x^2}$

Exam-style questions 1 page 29

1 a x^3 **b** $3x$

2 a $\frac{y}{\sqrt{x}}\times\frac{\sqrt{x}}{\sqrt{x}}=\frac{y\sqrt{x}}{x}$

b $\frac{3}{2-\sqrt{x}}\times\frac{2+\sqrt{x}}{2+\sqrt{x}}=\frac{6+3\sqrt{x}}{4-x}$

3 a $f(x)=2x^3+x^2-25x+12$

$f(3)=54+9-75+12=0$ so $(x-3)$ is a factor of $f(x)$.

b $f(-1)=-2+1+25+12=36$ so $(x+1)$ is not a factor of $f(x)$.

4 $256=2^8$

5 ${}^{10}C_3=\frac{10\times9\times8}{3\times2}=10\times3\times4=120$

6

$$\begin{array}{r}
x^2+x-2\\
(x-7)\overline{\big)\,x^3-6x^2-9x+14}\\
\underline{x^3-7x^2}\\
x^2-9x\\
\underline{x^2-7x}\\
-2x+14\\
\underline{-2x+14}\\
0
\end{array}$$

$x^3-6x^2-9x+14=(x-7)(x^2+x-2)$

$=(x-7)(x+2)(x-1)$

7 $\frac{125}{5\sqrt{5}}=\frac{5^3}{5^1\times5^{\frac{1}{2}}}=\frac{5^3}{5^{\frac{3}{2}}}=5^{\frac{3}{2}}$

8 $\left(a+\sqrt{b}\right)^2+\left(a-\sqrt{b}\right)^2$

$=\left(a+\sqrt{b}\right)\left(a+\sqrt{b}\right)+\left(a-\sqrt{b}\right)\left(a-\sqrt{b}\right)$

$=a^2+2a\sqrt{b}+b+a^2-2a\sqrt{b}+b$

$=2a^2+2b$

which is rational if a and b are rational.

9 a $\frac{\sqrt{x}-1}{\sqrt{x}+1}\times\frac{\sqrt{x}-1}{\sqrt{x}-1}=\frac{x-2\sqrt{x}+1}{x-1}$

b $\frac{2\sqrt{x}}{3-\sqrt{2x}}\times\frac{3+\sqrt{2x}}{3+\sqrt{2x}}=\frac{6\sqrt{x}+2x\sqrt{2}}{9-2x}$

10 $f(x)=x^4-9x^2-4x+12$

$f(1)=1-9-4+12=0$ hence $(x-1)$ is a factor of $f(x)$.

$$\begin{array}{r}
x^3+x^2-8x-12\\
(x-1)\overline{\big)\,x^4+0-9x^2-4x+12}\\
\underline{x^4-x^3}\\
x^3-9x^2\\
\underline{x^3-x^2}\\
-8x^2-4x\\
\underline{-8x^2+8x}\\
-12x+12\\
\underline{-12x+12}\\
0
\end{array}$$

$x^4-9x^2-4x+12=(x-1)(x^3+x^2-8x-12)$

$g(x)=x^3+x^2-8x-12$

$g(-2)=-8+4+16-12=0$ hence $(x+2)$ is a factor of $g(x)$.

$$\begin{array}{r}
x^2-x-6\\
(x+2)\overline{\big)\,x^3+x^2-8x-12}\\
\underline{x^3+2x^2}\\
-x^2-8x\\
\underline{-x^2-2x}\\
-6x-12\\
\underline{-6x-12}\\
0
\end{array}$$

$f(x)=x^4-9x^2-4x+12=(x-1)(x+2)^2(x-3)$

11 a $(4-3x)^5=4^5\left(1-\frac{4x}{3}\right)^4$

The index on the bracket should be 5 and the contents of the bracket should be $\left(1-\frac{3x}{4}\right)$.

b The coefficient of $x<0$ (it is -12).

12 $2x^3+15x^2+31x+12=(Ax^2+Bx+C)(x-5)+D$

$x=5$: $250+375+155+12=D$

$D=792$

$x = 0$: $12 = (C)(-5) + 792$

$C = 156$

Equating coefficients:

x^3: $2 = A$

x^2: $15 = -5A + B$

$B = 25$

$2x^3 + 15x^2 + 31x + 12 = (2x^2 + 25x + 156)(x - 5) + 792$

13 $\left(\frac{x}{27}\right)^{-\frac{2}{3}} = \left(\frac{27}{x}\right)^{\frac{2}{3}}$

$x^{\frac{2}{3}} = 16$

$\sqrt[3]{x^2} = 16$

$x^2 = 4096$

$x = \pm 64$

14 a $(a - 2x)^4 = a^4\left(1 - \frac{2x}{a}\right)^4$

For x:

$(a^4)(4)\left(-\frac{2x}{a}\right) = -216x$

$a^3 = 27$

$a = 3$

b $(3 - 2x)^4 = {}^4C_0(3)^4 + {}^4C_1(3)^3(-2x)^1 + {}^4C_2(3)^2(-2x)^2 + {}^4C_3(3)^1(-2x)^3 + {}^4C_4(-2x)^4$

$= \frac{4!}{4!0!}(81) + \frac{4!}{3!1!}(-54x) + \frac{4!}{2!2!}(36x^2) + \frac{4!}{1!3!}(-24x^3) + \frac{4!}{0!4!}(16x^4)$

$= 81 - 216x + 216x^2 - 96x^3 + 16x^4$

c $x = 0.1$

$(2.8)^4 = 81 - 21.6 + 2.16 - 0.096 + 0.0016$

$= 61.4656$

15 $\frac{12x^2 - 8x - 15}{42x^2 + 29x - 5} = \frac{(6x + 5)(2x - 3)}{(6x + 5)(7x - 1)}$

$= \frac{2x - 3}{7x - 1}$

16 $1 + 6 + 16 + 20 + 15 + 6 + 1 = 65 = 2^6 + 1$

So one number is 1 too high. As the numbers are symmetrical, the 16 should be a 15.

17 $(a + \sqrt{b}) \times (a - \sqrt{b})^3 = (a + \sqrt{b})(a - \sqrt{b})(a - \sqrt{b})(a - \sqrt{b})$

$= (a^2 - b)(a^2 - 2a\sqrt{b} + b)$

$= a^4 - 2a^3\sqrt{b} + a^2b - a^2b + 2ab\sqrt{b} - b^2$

$= a^4 - 2a^3\sqrt{b} + 2ab\sqrt{b} - b^2$

which is irrational unless b is a square number.

18 a $\frac{1}{2\sqrt{x} + 3} \times \frac{2\sqrt{x} - 3}{2\sqrt{x} - 3} + \frac{1}{\sqrt{x}} \times \frac{\sqrt{x}}{\sqrt{x}} = \frac{2\sqrt{x} - 3}{4x - 9} + \frac{\sqrt{x}}{x}$

$= \frac{2x\sqrt{x} - 3x + 4x\sqrt{x} - 9\sqrt{x}}{4x^2 - 9x}$

$= \frac{6x\sqrt{x} - 3x - 9\sqrt{x}}{4x^2 - 9x}$

b $\frac{3x}{\sqrt{x} - 1} \times \frac{\sqrt{x} + 1}{\sqrt{x} + 1} - \frac{\sqrt{x} + 1}{\sqrt{x}} \times \frac{\sqrt{x}}{\sqrt{x}} = \frac{3x\sqrt{x} + 3x}{x - 1} - \frac{x + \sqrt{x}}{x}$

$= \frac{3x^2\sqrt{x} + 3x^2 - x^2 - x\sqrt{x} + x + \sqrt{x}}{x^2 - x}$

$= \frac{3x^2\sqrt{x} + 2x^2 - x\sqrt{x} + x + \sqrt{x}}{x^2 - x}$

19 $f(x) = 4x^4 + 8x^3 - 21x^2 - 18x + 27$

$f(1) = 4 + 8 - 21 - 18 + 27 = 0$ so $(x - 1)$ is a factor of $f(x)$.

$$
\begin{array}{r}
4x^3 + 12x^2 - 9x - 27 \\
(x - 1)\overline{)\,4x^4 + 8x^3 - 21x^2 - 18x + 27} \\
\underline{4x^4 - 4x^3 \qquad\qquad\qquad\qquad} \\
12x^3 - 21x^2 \qquad\qquad \\
\underline{12x^3 - 12x^2 \qquad\qquad} \\
-9x^2 - 18x \qquad \\
\underline{-9x^2 + 9x \qquad} \\
-27x + 27 \\
\underline{-27x + 27} \\
0
\end{array}
$$

$x^4 - 9x^2 - 4x + 12 = (x - 1)(4x^3 + 12x^2 - 9x - 27)$

$g(x) = 4x^3 + 12x^2 - 9x - 27$

$g(-3) = -108 + 108 + 27 - 27 = 0$ so $(x + 3)$ is a factor of $g(x)$.

$$
\begin{array}{r}
4x^2 - 9 \\
(x + 3)\overline{)\,4x^3 + 12x^2 - 9x - 27} \\
\underline{4x^3 + 12x^2 \qquad\qquad} \\
-9x - 27 \\
\underline{-9x - 27} \\
0
\end{array}
$$

$f(x) = 4x^4 + 8x^3 - 21x^2 - 18x + 27$

$= (x - 1)(x + 3)(2x - 3)(2x + 3)$

20 $2x^5 - 19x^4 + 60x^3 - 65x^2 - 2x + 24$

$= (Ax^4 + Bx^3 + Cx^2 + Dx + E)(x + 2) + F$

$x = -2$: $-64 - 304 - 480 - 260 + 4 + 24 = F$

$F = -1080$

$x = 0$: $24 = (E)(2) - 1080$

$E = 552$

Equating coefficients:

x^5: $2 = A$

x^4: $-19 = 2A + B$

$B = -23$

x^3: $60 = 2B + C$

$C = 106$

x^2: $-65 = 2C + D$

$D = -227$

$2x^5 - 19x^4 + 60x^3 - 65x^2 - 2x + 24$
$= (2x^4 - 23x^3 + 106x^2 - 227x + 552)(x + 2) - 1080$

21 a $\left(1+\sqrt{x}\right)^3 = {}^3C_0 + {}^3C_1\left(\sqrt{x}\right)^1 + {}^3C_2\left(\sqrt{x}\right)^2 + {}^3C_3\left(\sqrt{x}\right)^3$

$= \frac{3!}{3!0!} + \frac{3!}{2!1!}\left(\sqrt{x}\right) + \frac{3!}{1!2!}(x) + \frac{3!}{0!3!}\left(x\sqrt{x}\right)$

$= 1 + 3\sqrt{x} + 3x + x\sqrt{x}$

b $\left(2-2\sqrt{x}\right)^3 = 2^3\left(1-\sqrt{x}\right)^3 = 8 - 24\sqrt{x} + 24x - 8x\sqrt{x}$

22 Total number of balls is $(7x - 3) + (9 - x) = 6x + 6$

$\mathrm{P}(RR) = \frac{7x-3}{6x+6} \times \frac{7x-4}{6x+5}$

$= \frac{49x^2 - 49x + 12}{36x^2 + 66x + 30}$

23 a $\left(1+\frac{x}{2}\right)^6 = {}^6C_0 + {}^6C_1\left(\frac{x}{2}\right)^1 + {}^6C_2\left(\frac{x}{2}\right)^2 + {}^6C_3\left(\frac{x}{2}\right)^3 + \ldots$

$= \frac{6!}{6!0!} + \frac{6!}{5!1!}\left(\frac{x}{2}\right) + \frac{6!}{4!2!}\left(\frac{x^2}{4}\right) + \frac{6!}{3!3!}\left(\frac{x^3}{8}\right) + \ldots$

$= 1 + 3x + \frac{15x^2}{4} + \frac{5x^3}{2} + \ldots$

b $x = 0.1$

$1.05^6 \approx 1 + 0.3 + 0.0375 + 0.0025$

≈ 1.34

c It is only an approximation because the expansion isn't complete.

24

$$
\begin{array}{r}
x^2 + x - 6 \\
2x+1 \overline{\big)\, 2x^3 + 3x^2 - 11x - 7} \\
\underline{2x^3 + x^2} \qquad\qquad \\
2x^2 - 11x \qquad \\
\underline{2x^2 + x} \qquad \\
-12x - 7 \\
\underline{-12x - 6} \\
1
\end{array}
$$

When $2x^3 + 3x^2 - 11x - 7$ is divided by $2x + 1$ to get an expression for the area of the cross-section (width × depth), then there is a remainder of $\frac{1}{2x+1}$. Consequently, this is not a possible correct expression for the volume of the cuboid.

When $2x^3 + 3x^2 - 11x - 6$ is divided by $2x + 1$ to get an expression for the area of the cross-section (width × depth) then there is no remainder. Consequently, this is a possible correct expression for the volume of the cuboid.

2 Algebra and functions 2: Equations and inequalities

Prior knowledge

page 32

1 $2x^2 - 5x - 3 = 0$

$(2x + 1)(x - 3) = 0$

$x = -\frac{1}{2}$ or $x = 3$

2 $x^2 + x - 12 = 0$

$\left(x + \frac{1}{2}\right)^2 - \frac{1}{4} - 12 = 0$

$\left(x + \frac{1}{2}\right)^2 = \frac{49}{4}$

$x = 3$ or -4

3 $a = 2$, $b = -8$, $c = -13$

$b^2 - 4ac = (-8)^2 - (4)(2)(-13) = 168$ so two real roots.

$x = \frac{-(-8) \pm \sqrt{168}}{(2)(2)}$

$x = \frac{8 \pm \sqrt{168}}{4}$

$x = 5.240$ or -1.24

4 $5x - y = 13$ ①

$2x + y = 1$ ②

Add ① and ②.

$7x = 14$

$x = 2$, $y = -3$

5 $y = x + 1$ ①

$y = x^2 - 3x + 2$ ②

Substitute ① into ②.

$x + 1 = x^2 - 3x + 2$

$x^2 - 4x + 1 = 0$

$(x - 2)^2 - 4 + 1 = 0$

$(x - 2)^2 = 3$

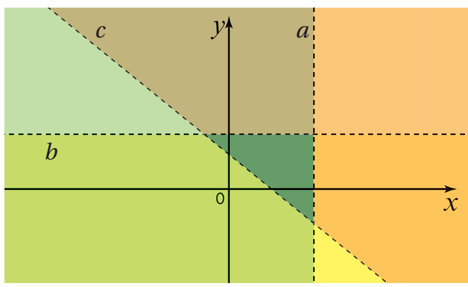
$x = 2 + \sqrt{3}$ or $2 - \sqrt{3}$

6 $3x - 5 < 7$

$3x < 12$

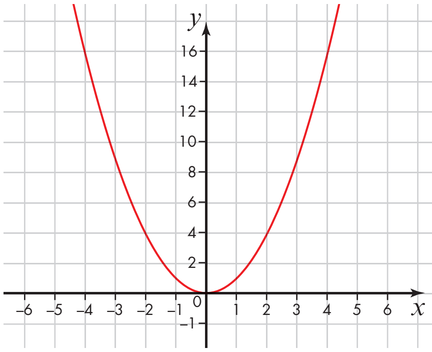
$x < 4$

7

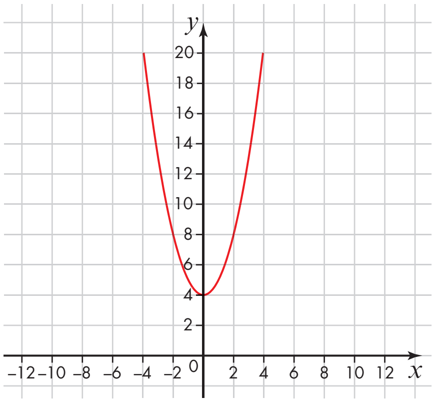


Exercise 2.1A page 34

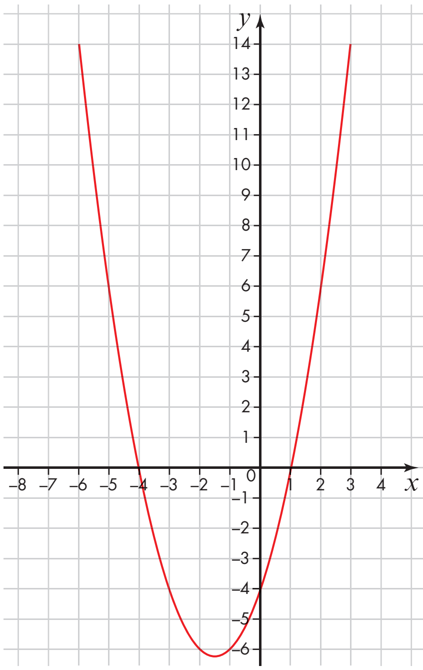
1 **a** $y = x^2$ where $-4 \leqslant x \leqslant 4$. Intercepts and turning point at (0, 0).



b No x-intercept, y-intercept and turning point at (0, 4).

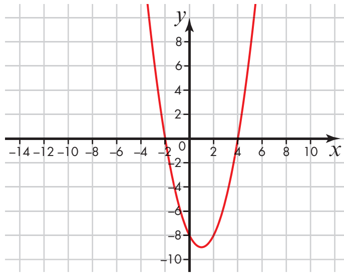


c $y = x^2 + 3x - 4$ where $-5 \leqslant x \leqslant 3$. x-intercepts at (–4, 0) and (1, 0), y-intercept at (0, –4) and turning point at $\left(-\frac{3}{2}, -\frac{25}{4}\right)$.

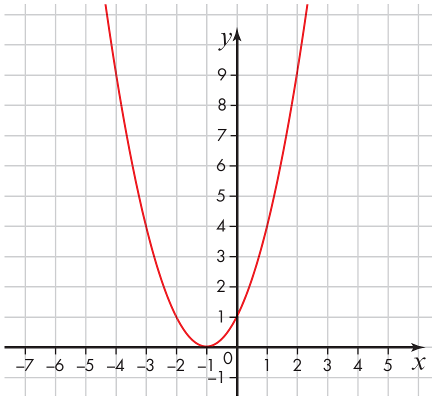


2

x	–3	–2	–1	0	1	2	3	4	5
x^2	**9**	4	1	0	1	4	9	16	25
$-2x$	6	**4**	2	0	–2	–4	–6	–8	–10
–8	–8	–8	–8	–8	–8	–8	–8	**–8**	–8
y	**7**	**0**	–5	–8	–9	–8	–5	**0**	7

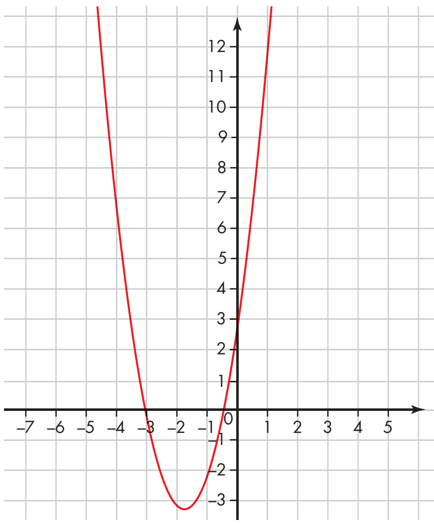


3 **a** Line of symmetry at $x = -1$, y-intercept at (0, 1), x-intercept and turning point at (–1, 0).



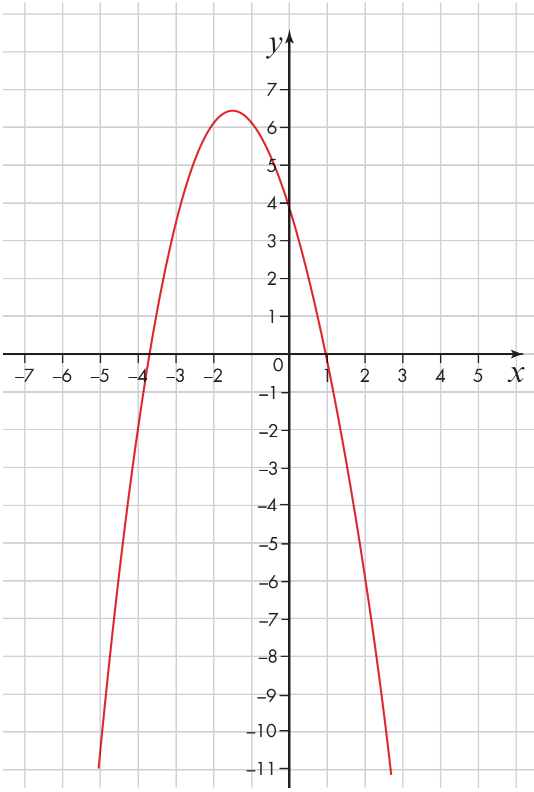
b Line of symmetry at $x = -1.75$.

x-intercepts at (–3, 0) and $(-\frac{1}{2}, 0)$, y-intercept at (0, 3) and turning point at (–1.75, –3.1).



c Line of symmetry at $x = -1.5$.

x-intercepts at (–4, 0) and (1, 0), y-intercept at (0, 4) and turning point at (–1.5, 6.3).



4 Line of symmetry at $x = 1.2$ (approx.).

x-intercepts at (–0.7, 0) and (3, 0), y-intercept at (0, –6) and turning point at approx. (1.2, –10).

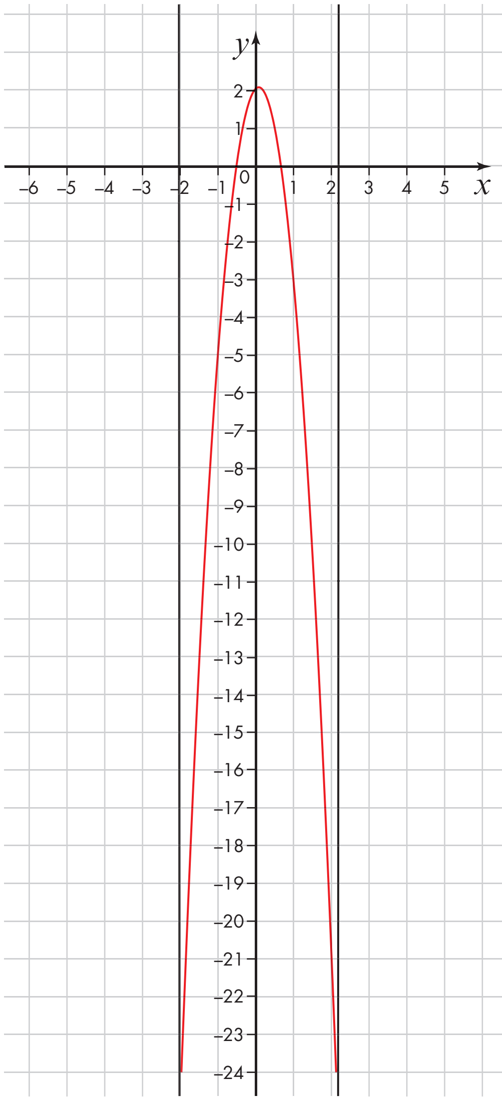
$(3x + 2)(x - 3) = 0$

Expand the brackets.

$$y = 3x^2 - 7x - 6$$

This equation has the approximate x-intercepts and the y-intercept at the turning point when $x = 1.2$, $y = -10.08$ in the equation.

5 $y = 2 + x - 6x^2$



$98 - x - 24x^2 = 0$

Rearranging.

$2 + x - 6x^2 = 18x^2 + 2x - 96$

Solutions for $98 - x - 24x^2 = 0$ can be found where the graphs intersect, hence $x = -2$ and $x = 2$.

Exercise 2.2A page 36

1 a $x^2 + 2x + 13 = 0$

$b^2 - 4ac = (2)^2 - (4)(1)(13)$

$= -48$

$b^2 - 4ac < 0$ so no real roots.

b $2x^2 + 3x + 1 = 0$

$b^2 - 4ac = (3)^2 - (4)(2)(1)$

$= 1$

$b^2 - 4ac > 0$ so two distinct real roots.

c $x^2 + 4x + 4 = 0$

$b^2 - 4ac = (4)^2 - (4)(1)(4)$

$= 0$

$b^2 - 4ac = 0$ so two equal real roots.

2 $2x^2 - 5x + 3 = 0$

$b^2 - 4ac = (-5)^2 - (4)(2)(3)$

$= 25 - 24$

$= 1$

$b^2 - 4ac > 0$ so has two **distinct** real roots.

3 a $x^2 - 3x - 5 = 0$

$b^2 - 4ac = (-3)^2 - (4)(1)(-5)$

$= 29$

$b^2 - 4ac > 0$ so two distinct real roots.

b $3x^2 - 2x + 7 = 0$

$b^2 - 4ac = (-2)^2 - (4)(3)(7)$

$= -80$

$b^2 - 4ac < 0$ so no real roots.

c $x^2 - 12x + 36 = 0$

$b^2 - 4ac = (-12)^2 - (4)(1)(36)$

$= 0$

$b^2 - 4ac = 0$ so two equal real roots.

4 $x^2 + (2k + 2)x + (5k - 1) = 0$

$b^2 - 4ac = 0$ so two equal real roots.

$(2k + 2)^2 - (4)(1)(5k - 1) = 0$

Expand and simplify.

$k^2 - 3k + 2 = 0$ *or* $-k^2 + 3k - 2 = 0$

Both quadratic equations have the solutions $k = 1$ or $k = 2$. For $k = 1$, $x = -2$ and for $k = 2$, $x = -3$.

5 a $4 + 2x - 3x^2 = 0$

$b^2 - 4ac = (2)^2 - (4)(-3)(4)$

$= 52$

$b^2 - 4ac > 0$ so two distinct real roots.

b $49 + x^2 - 14x = 0$

$b^2 - 4ac = (-14)^2 - (4)(1)(49)$

$= 0$

$b^2 - 4ac = 0$ so two equal real roots.

c $3x^2 + 12 - 5x = 0$

$b^2 - 4ac = (-5)^2 - (4)(3)(12)$

$= -119$

$b^2 - 4ac < 0$ so no real roots.

6 $2x^2 + 5x - c = 0$

a $b^2 - 4ac < 0$

$(5)^2 - (4)(2)(-c) < 0$

$25 + 8c < 0$

$c < -\frac{25}{8}$

b $b^2 - 4ac = 0$

$(5)^2 - (4)(2)(-c) = 0$

$25 + 8c = 0$

$c = -\frac{25}{8}$

c $b^2 - 4ac > 0$

$(5)^2 - (4)(2)(-c) > 0$

$25 + 8c > 0$

$c > -\frac{25}{8}$

Exercise 2.3A page 38

1 a $x^2 + 4x = (x + 2)^2 - 4$

b $2x^2 - 8x = 2(x^2 - 4x)$

$= 2[(x - 2)^2 - 4]$

c $x^2 + 8x + 7 = (x + 4)^2 - 16 + 7$

$= (x + 4)^2 - 9$

2 $x^2 - 10x + 11 = (x - 5)^2 - 25 + 11$

$= (x - 5)^2 - 14$

3 a $x^2 - 8x - 5 = (x - 4)^2 - 16 - 5$

$= (x - 4)^2 - 21$

b $x^2 + 3x - 7 = \left(x + \frac{3}{2}\right)^2 - \frac{9}{4} - 7$

$= \left(x + \frac{3}{2}\right)^2 - \frac{37}{4}$

c $2x^2 + 3x + 9 = 2\left(x^2 + \frac{3x}{2} + \frac{9}{2}\right)$

$= 2\left[\left(x + \frac{3}{4}\right)^2 - \frac{9}{16} + \frac{9}{2}\right]$

$= 2\left[\left(x + \frac{3}{4}\right)^2 + \frac{63}{16}\right]$

4 $x^2 - 5x + 7 = \left(x - \frac{5}{2}\right)^2 - \frac{25}{4} + 7$

$y = \left(x - \frac{5}{2}\right)^2 + \frac{3}{4}$

At the turning point y has a minimum value, i.e. when $x = \frac{5}{2}$. When $x = \frac{5}{2}, y = \frac{3}{4}$, the coordinates of the turning point.

5 $2[(x+2)^2 - \frac{13}{2}] = 2(x^2 + 4x + 4 - \frac{13}{2})$

$= 2x^2 + 8x + 8 - 13$

$= 2x^2 + 8x - 5$

6 **a** $5x^2 + \frac{1}{2}x - 3 = 5\left(x^2 + \frac{x}{10} - \frac{3}{5}\right)$

$= 5\left[\left(x + \frac{1}{20}\right)^2 - \frac{1}{400} - \frac{3}{5}\right]$

$= 5\left[\left(x + \frac{1}{20}\right)^2 - \frac{241}{400}\right]$

b $5x^2 - 18 = 5\left(x^2 - \frac{18}{5}\right)$ but can go no further.

c $5 - 7x - 3x^2 = -3\left(x^2 + \frac{7x}{3} - \frac{5}{3}\right)$

$= -3\left[\left(x + \frac{7}{6}\right)^2 - \frac{49}{36} - \frac{5}{3}\right]$

$= -3\left[\left(x + \frac{7}{6}\right)^2 - \frac{109}{36}\right]$

7 $2x^2 - 3x + 11 = 2\left(x^2 - \frac{3x}{2} + \frac{11}{2}\right)$

$= 2\left[\left(x - \frac{3}{4}\right)^2 - \frac{9}{16} + \frac{11}{2}\right]$

$= 2\left[\left(x - \frac{3}{4}\right)^2 + \frac{79}{16}\right]$

$2\left[\left(x - \frac{3}{2}\right)^2 + \frac{79}{16}\right] > 0$

Consequently $b^2 - 4ac < 0$ because the curve of this equation will not intercept the x-axis and so there are no real roots.

8 $(1-2x)[(x+4)^2 - 1] = (1-2x)(x^2 + 8x + 16 - 1)$

$= (1-2x)(x^2 + 8x + 15)$

$= x^2 + 8x + 15 - 2x^3 - 16x^2 - 30x$

$= 15 - 22x - 15x^2 - 2x^3$

Exercise 2.4A page 41

1 **a** **i** $x^2 - 9 = 0$

$(x+3)(x-3) = 0$

$x = 3$ or -3

ii $x = \dfrac{0 \pm \sqrt{0 - (4)(1)(-9)}}{2}$

$x = 3$ or -3

iii $x^2 = 9$

$x = 3$ or -3

iv $x = 3$ or -3

b **i** $x^2 + 3x + 2 = 0$

$(x+1)(x+2) = 0$

$x = -1$ or -2

ii $x = \dfrac{-3 \pm \sqrt{9 - (4)(1)(2)}}{2}$

$x = -1$ or -2

iii $\left(x + \frac{3}{2}\right)^2 - \frac{1}{4} = 0$

$x = -1$ or -2

iv $x = -1$ or -2

c **i** $x^2 - 5x - 7 = 0$

Cannot be factorised.

ii $x = \dfrac{5 \pm \sqrt{25 - (4)(1)(-7)}}{2}$

$x = \dfrac{5 \pm \sqrt{53}}{2}$

iii $\left(x - \frac{5}{2}\right)^2 - \frac{53}{4} = 0$

$x = \dfrac{5 \pm \sqrt{53}}{2}$

iv $x = \dfrac{5 \pm \sqrt{53}}{2}$

d **i** $x(x-6) = 0$

$x = 0$ or 6

ii $x = \dfrac{6 \pm \sqrt{36 - (4)(1)(0)}}{2}$

$x = 0$ or 6

iii $(x-3)^2 - 9 = 0$

$x = 0$ or 6

iv $x = 0$ or 6

2 **Factorisation:**

- can only be used on equations that factorise
- sometimes spotting factors can be difficult
- can solve: $x^2 + 3x + 2 = 0$
- cannot solve: $x^2 - 5x - 7 = 0$

Quadratic formula:

- can be used to solve any equation with real roots including ones that don't factorise
- cumbersome and consequently easy to make a mistake
- can solve: $x^2 - 5x - 7 = 0$
- cannot solve: $x^2 - 5x + 7 = 0$

Completing the square:

- can be used to solve any equation with real roots including ones that don't factorise
- can be cumbersome manipulations if b is odd and $a > 0$; consequently easy to make a mistake
- can solve: $x^2 - 5x - 7 = 0$
- cannot solve: $x^2 - 5x + 7 = 0$

Calculator:

- easy if you know how to use your calculator
- does not help you to understand the underlying methods
- can solve: $x^2 + 3x + 2 = 0$ and $x^2 - 5x - 7 = 0$
- cannot solve: $x^2 - 5x + 7 = 0$

3 a $(x - 4)^2 = 13$

$x = 4 \pm \sqrt{13}$

Completed the square. The equation was already in form of a completed square.

b $2x^2 - 5x - 11 = 0$

$$x = \frac{5 \pm \sqrt{25 - (4)(2)(-11)}}{4}$$

$$x = \frac{5 \pm \sqrt{113}}{4}$$

Does not factorise so used the quadratic formula. Alternatively, could have completed the square but chose not to as $a > 1$.

c $3x^2 = 5 - 14x$

$3x^2 + 14x - 5 = 0$

$(3x - 1)(x + 5) = 0$

$x = \frac{1}{3}$ or -5

Equation factorises so easy to do this. Alternatively, could have completed the square but chose not to as $a > 1$.

d $4x^2 - 16x + 7 = 0$

$(2x - 1)(2x - 7) = 0$

$x = \frac{1}{2}$ or $\frac{7}{2}$

Equation factorises so easy to do this. Alternatively, could have completed the square but chose not to as $a > 1$.

4 a Cannot be factorised if $b^2 - 4ac$ is not a square number.

$b^2 - 4ac = 16 - (4)(3)(-11) = 148$

148 not a square number so does not factorise.

b $3x^2 + 4x - 11 = 0$

$$x = \frac{-4 \pm \sqrt{148}}{6}$$

$$x = \frac{-2 \pm \sqrt{37}}{3}$$

5 $(x - 3)(x - 5) = 8$

$x^2 - 8x + 15 - 8 = 0$

$x^2 - 8x + 7 = 0$

$(x - 7)(x - 1) = 0$

$x = 1$ or 7

6 $ax^2 + bx + c = 0$

$$a\left(x^2 + \frac{bx}{a} + \frac{c}{a}\right) = 0$$

$$\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a} = 0$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{4ac}{4a^2}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$$x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Exercise 2.5A **page 43**

1 a $2x + y = -4$ ①

$5x + y = -1$ ②

Subtract ① from ②.

$3x = 3$

$x = 1$ and $y = -6$

b $3x - 2y = 2$ ①

$5x + y = 25$ ②

Multiply ② by 2.

$10x + 2y = 50$ ③

Add ① and ③.

$13x = 52$

$x = 4$ and $y = 5$

c $x - 2y = 13$ ①

$-x + 3y = -15$ ②

Add ① and ②.

$y = -2$ and $x = 9$

2 $5x + 6y = 6$ ①
$3x - y = 22$ ②
Multiply ② by 6.
$5x + 6y = 6$ ①
$18x - 6y = 132$ ③
Add ① and ③.
$23x = 138$
$x = 6$
Substitute x into ②.
$\mathbf{18} - y = 22$
$y = \mathbf{-4}$

3 If there are n unknowns then you need n distinct equations involving the n unknowns.

4 $2x + y = -1$ ①
$3x - 2y = -12$ ②
Subtract ② from ①.
$x - 3y = -11$
As neither the coefficient of x nor y is the same in both equations, when one equation is subtracted from the other an unknown is not eliminated.

5 $3x - 4y = 3$ ①
$6x + 4y = 3$ ②
Add ① and ②.
$9x = 6$
$x = \frac{2}{3}$ and $y = -\frac{1}{4}$

6 $x + y = 50$ ①
$x - 1 = 15(y - 1)$
$x - 15y = -14$ ②
Subtract ② from ①.
$16y = 64$
$y = 4$ and $x = 46$. 46 years old.

7 $2x - y = -14$ ①
$3y - 2z = 16$ ②
$z - x = 3$ ③
Multiply ① by 3.
$6x - 3y = -42$ ④
$3y - 2z = 16$ ②
Add ④ and ②.
$6x - 2z = -26$ ⑤
Multiply ③ by 2.
$2z - 2x = 6$ ⑥
Add ⑤ and ⑥.
$4x = -20$
$x = -5$
Substitute x into ③: $z = -2$
Substitute x into ①: $y = 4$

Exercise 2.5B

page 45

1 a $2x + y = -4$ ①
$5x + y = -1$ ②
From ②.
$y = -1 - 5x$
Substitute into ①.
$2x - 1 - 5x = -4$
$x = 1$ and $y = -6$

b $3x - 2y = 2$ ①
$5x + y = 25$ ②
From ②.
$y = 25 - 5x$
Substitute into ①.
$3x - 2(25 - 5x) = 2$
$x = 4$ and $y = 5$

c $x - 2y = 13$ ①
$-x + 3y = -15$ ②
From ①.
$x = 13 + 2y$
Substitute into ②.
$-13 - 2y + 3y = -15$
$y = -2$ and $x = 9$

2 $5x + 6y = 6$ ①
$3x - y = 22$ ②
Rearrange ②.
$3x - 22 = y$
Substitute into ①.
$5x + 6(3x - 22) = 6$
Expand and simplify.
$23x - 132 = 6$
$x = 6$
Substitute x into ②.
$\mathbf{18} - 22 = y$
Solve.
$y = -4$

3 $4x + 3y = 9$ ①

$5x - 6y = 60$ ②

From ①.

$$x = \frac{9 - 3y}{4}$$

Substitute into ②.

$$5\left(\frac{9-3y}{4}\right) - 6y = 60$$

$y = -5$ and $x = 6$

The graphs intersect at the point (6, –5).

4 $3x - 4y = 3$ ①

$6x + 4y = 3$ ②

From ①.

$$x = \frac{3 + 4y}{3}$$

Substitute into ②.

$$6\left(\frac{3+4y}{3}\right) + 4y = 3$$

$y = -\frac{1}{4}$ and $x = \frac{2}{3}$

5 $2c + 7b = 170$ ①

$3c + 8b = 250$ ②

From ①.

$$c = \frac{170 - 7b}{2}$$

Substitute into ②.

$$3\left(\frac{170-7b}{2}\right) + 8b = 250$$

$b = 2$p and $c = 78$p

The simultaneous equations only work if the cost of a small bag of sweets is 78p and the cost of a bar of chocolate is 2p. Although two equations can be formed and solved simultaneously from the given figures, 2p is too little for a bar of chocolate so the student must be correct: one of the calculations is wrong.

6 $2x - y = -14$ ①

$3y - 2z = 16$ ②

$z - x = 3$ ③

From ①.

$2x + 14 = y$

Substitute into ②.

$3(2x + 14) - 2z = 16$

$6x - 2z = -26$ ④

From ③.

$z = x + 3$

Substitute into ④.

$6x - 2(x + 3) = -26$

$4x = -20$

$x = -5$, $z = -2$ and $y = 4$

Exercise 2.6A — page 47

1 a $x + y = 3$ ①

$2x^2 - y = 25$ ②

Add ① and ②.

$2x^2 + x - 28 = 0$

$(2x - 7)(x + 4) = 0$

$x = \frac{7}{2}$ or -4

$y = -\frac{1}{2}$ or 7

b $2x - y = 20$ ①

$x^2 + xy = -12$ ②

From ①.

$2x - 20 = y$

Substitute into ②.

$x^2 + x(2x - 20) = -12$

$3x^2 - 20x + 12 = 0$

$(3x - 2)(x - 6) = 0$

$x = \frac{2}{3}$ or 6

$y = -\frac{56}{3}$ or -8

c $y = 4x$ ①

$5 - x^2 = y$ ②

① = ②

$5 - x^2 = 4x$

$x^2 + 4x - 5 = 0$

$(x + 5)(x - 1) = 0$

$x = -5$ or 1

$y = -20$ or 4

2 a Substitution or elimination: for elimination the second equation would need to be multiplied by 2 so that y can subsequently be eliminated.

b Substitution only: neither addition nor subtraction of the equations will eliminate a variable.

c Substitution only: neither addition nor subtraction of the equations will eliminate a variable.

3 a $2x^2 + y = 14$ ①

$x - 2y = 11$ ②

Multiply ① by ②.

$4x^2 + 2y = 28$ ③

Add ② and ③.

$4x^2 + x - 39 = 0$

$(4x + 13)(x - 3) = 0$

$x = -\frac{13}{4}$ or 3

$y = -\frac{57}{8}$ or -4

b $xy - x = -4$ ①

$x + y = 1$ ②

From ②.

$y = 1 - x$

Substitute into ①.

$x(1 - x) - x = -4$

$x - x^2 - x = -4$

$x = \pm 2$ $y = -1$ or 3

c $x - y = 10$ ①

$xy = 140$ ②

From ①.

$x = y + 10$

Substitute into ②.

$y(y + 10) = 140$

$y^2 + 10y - 140 = 0$

$$y = \frac{-10 \pm \sqrt{100 - (4)(1)(-140)}}{2}$$

$$y = \frac{-10 \pm \sqrt{660}}{2}$$

$y = -5 \pm \sqrt{165}$ and $x = 5 \pm \sqrt{165}$

4 $2x - 3y = 13$ ①

$x^2 - y = 7$ ②

From ②.

$x^2 - 7 = y$

Substitute into ①.

$2x - 3(x^2 - 7) = 13$

$3x^2 - 2x - 8 = 0$

$(3x + 4)(x - 2) = 0$

$x = -\frac{4}{3}$ or 2

$y = -\frac{47}{9}$ or -3

So the coordinates of the points of intersection of this line and this curve are $(-\frac{4}{3}, -\frac{47}{9})$ and $(2, -3)$.

5 $y - x = 1$ ①

$x^2 + y^2 = 64$ ②

From ①.

$y = x + 1$

Substitute into ②.

$x^2 + (x + 1)^2 = 64$

$2x^2 + 2x - 63 = 0$

$$x = \frac{-2 \pm \sqrt{4 - (4)(2)(-63)}}{4}$$

$x = 5.135$ or -6.135

$y = 6.135$ or -5.135

So the coordinates of the points of intersection of this line and the circle are (5.135, 6.135) and (−6.135, −5.135).

6 $y - x = 10$ ①

$x^2 + y^2 = 50$ ②

From ①.

$y = x + 10$

Substitute into ②.

$x^2 + (x + 10)^2 = 50$

$2x^2 + 20x + 50 = 0$

$x^2 + 10x + 25 = 0$

$(x + 5)^2 = 0$

$x = -5$ and $y = 5$

The line is a tangent to the circle. They intersect once at (−5, 5).

Exercise 2.7A

page 51

1 **a** $4x - 3 > 5$

$4x > 8$

$x > 2$

b $3x + 7 \leqslant 28$

$3x \leqslant 21$

$x \leqslant 7$

c $5x + 6 < x - 2$

$4x < -8$

$x < -2$

d $3(x - 2) \geqslant 2 - 4x$

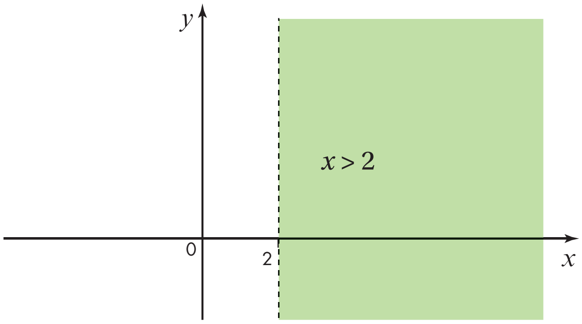
$3x - 6 \geqslant 2 - 4x$

$7x \geqslant 8$

$x \geqslant \frac{8}{7}$

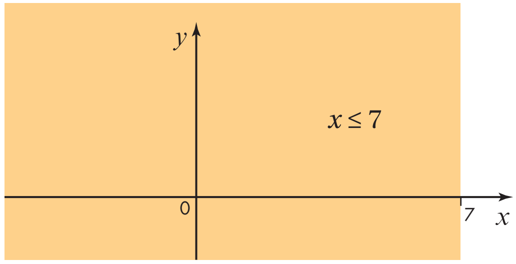
2 **a** $4x - 3 > 5$

$x > 2$



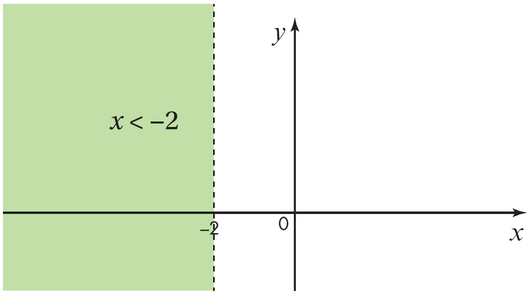
b $3x + 7 \leqslant 28$

$x \leqslant 7$



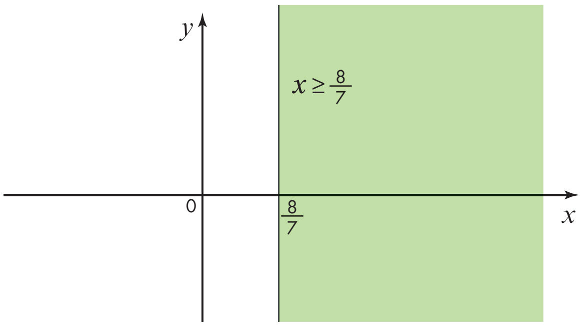
c $5x + 6 < x - 2$

$x < -2$



d $3(x - 2) \geqslant 2 - 4x$

$x \geqslant \frac{8}{7}$



3 $r \leqslant 10$

$2\pi r \geqslant 55$

$r \geqslant 8.8$

So $8.8 \leqslant r \leqslant 10$ miles

4 a $4(3x - 2) \geqslant 2x + 2$

$12x - 8 \geqslant 2x + 2$

$10x \geqslant 10$

$x \geqslant 1$

and $11 - 2x > \frac{x}{5}$

$55 - 10x > x$

$55 > 11x$

$x < 5$

$1 \leqslant x < 5$

$\{1, 2, 3, 4\}$

b $7 \leqslant 2x - 1 < 15$

$8 \leqslant 2x < 16$

$4 \leqslant x < 8$

or $-2 < \frac{x + 3}{4} < 2$

$-8 < x + 3 < 8$

$-11 < x < 5$

$-11 < x < 8$

$\{-10, -9, -8, -7, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7\}$

c $2(x - 3) \leqslant 4x - 12$

$2x - 6 \leqslant 4x - 12$

$6 \leqslant 2x$

$2x \geqslant 6$

$x \geqslant 3$

and $6x - 5 < -(2 + x)$

$6x - 5 < -2 - x$

$7x < 3$

$x < \frac{3}{7}$

x cannot be both less than $\frac{3}{7}$ and greater than or equal to 3.

d $2 - 9x \geqslant 7x$

$2 \geqslant 16x$

$x \leqslant \frac{1}{8}$

or $85 - 15x \geqslant x + 5$

$80 \geqslant 16x$

$x \leqslant 5$ where $x \geqslant 0$

$x \leqslant 5$

$\{0, 1, 2, 3, 4, 5\}$

5 a x cannot be both less than –2 and greater than 3 in this way.

Can be rewritten as $x < -2$ **or** $x > 3$

b x cannot be both less than –2 or greater than 3 in this way.

Can be rewritten as $x < -2$ **or** $x > 3$

c x cannot be both less than –2 and greater than 3.

6 $y \leqslant 5$, $y \geqslant 2x - 1$ and $y \leqslant 5 - x$

7 **a** $x \leqslant 3$ **b** $x \geqslant 3$

c $x \leqslant -3$ **d** $x \geqslant -3$

8 $4600 + 5j < 7000$

$5j < 2400$

$j < 480$

$5750 + 3j > 7000$

$j > 417$

$417 < j < 480$

Exercise 2.8A page 55

1 **a** $x^2 + 3x - 4 > 0$

$x^2 + 3x - 4 > 0$

$(x + 4)(x - 1) > 0$

$x < -4$ or $x > 1$

b $x^2 - 6x + 8 \leqslant 0$

$(x - 4)(x - 2) \leqslant 0$

$2 \leqslant x \leqslant 4$

c $x^2 - 9 \geqslant 0$

$(x + 3)(x - 3) \geqslant 0$

$x < -3$ or $x > 3$

d $x^2 - 6x < 0$

$x(x - 6) < 0$

$0 < x < 6$

2 **a** $2x^2 + 7x < -3$

$2x^2 + 7x + 3 < 0$

$(2x + 1)(x + 3) < 0$

$-3 < x < -\frac{1}{2}$

b $-x^2 - 3x + 4 > 0$

$(-x + 1)(x + 4) > 0$

$-4 < x < 1$

c $x^2 > 4$

$x^2 - 4 > 0$

$(x - 2)(x + 2) > 0$

$x < -2$ or $x > 2$

d $3x^2 \leqslant 5x$

$3x^2 - 5x \leqslant 0$

$x(3x - 5) \leqslant 0$

$0 < x < \frac{5}{3}$

3 **a** $6 - 5x - x^2 > 0$

$(1 - x)(x + 6) > 0$

$-6 < x < 1$

and $-x + 6 > 0$

$6 > x$

$x < 6$

$\{-5, -4, -3, -2, -1, 0\}$

b $2x^2 + 9x - 5 \leqslant 0$

$(2x - 1)(x + 5) \leqslant 0$

$-5 \leqslant x \leqslant \frac{1}{2}$

or $2x < 5$

$x < \frac{5}{2}$

So $x < \frac{5}{2}$

{all integers less than 2.5}

c $(2x + 1)^2 - 9 \leqslant 0$

$2x + 1 \leqslant \pm 3$

$-2 \leqslant x \leqslant 1$

and $2x < 6$

$x < 3$

$-2 \leqslant x \leqslant 1$

$\{-2, -1, 0, 1\}$

d $4x^2 < 3x$

$4x^2 - 3x < 0$

$x(4x - 3) < 0$

$0 < x < \frac{3}{4}$

or

$\frac{1 - 4x}{4} > 0$

$1 > 4x$

$x < \frac{1}{4}$

So $x < \frac{5}{2}$

{all integers less than 0.25}

4 $r \leqslant 10$

$\pi r^2 \geqslant 250$

$r \geqslant 8.9$

So $8.9 \leqslant r \leqslant 10$ miles

5 **a** $x^2 + 4 < 7x - 2$

$x^2 - 7x + 6 < 0$

$(x - 1)(x - 6) < 0$

$1 < x < 6$

$\{2, 3, 4, 5\}$

b $2x^2 - 3x - 15 < 2x - 3$

$2x^2 - 5x - 12 < 0$

$(2x+3)(x-4)<0$

$-\frac{3}{2}<x<4$

$\{-1, 0, 1, 2, 3\}$

c $12+9x+x^2>3(x+1)$

$12+9x+x^2>3x+3$

$9+6x+x^2>0$

$(3+x)^2>0$

Inequality true for all values of x.

d $x+1>3-4x-3x^2$

$0>2-5x-3x^2$

$3x^2+5x-2>0$

$(3x-1)(x+2)>0$

$x<-2$ or $x>\frac{1}{3}$

{all integers except –2, –1, 0}

6 a $x^2+5x-6<0$

$(x+6)(x-1)<0$

$-6<x<1$

and $x^2+3x-4<0$

$(x+4)(x-1)<0$

$-4<x<1$

So $-4<x<1$, i.e. $\{-3, -2, -1, 0\}$

b $x^2+5x<6$ or $x^2+3x<4$

So $-6<x<1$, i.e. $\{-5, -4, -3, -2, -1, 0\}$

c $x^2-6<-5x$ and $x^2-4>-3x$

$-6<x<1$ and $x<-4$ or $x>1$

So $-6<x<-4$, i.e. $\{-5\}$

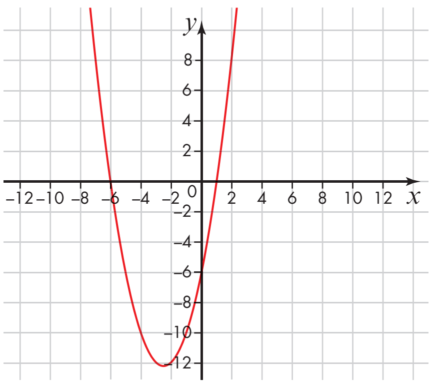
d $x^2+5x-6>0$ or $x^2+3x-4<0$

$x<-6$ or $x>1$ or $-4<x<1$

{all integers excluding –6, –5, –4, 1}

Exam-style questions 2 page 56

1 $y=x^2+5x-6$



Line of symmetry at $x=-2.5$

x-intercepts at (–6, 0) and (1, 0), y-intercept at (0, –6) and turning point at (–2.5, –12.3) (approx.).

2 $y=2x^2-3x-7$

$b^2-4ac=9-(4)(2)(-7)=65>0$ so two distinct real roots

3 $x=\dfrac{3\pm\sqrt{9-(4)(3)(-11)}}{6}$

$x=\dfrac{3\pm\sqrt{141}}{6}$

$x=-1.479$ or $x=2.479$

$3x^2-3x-11>0$

$x<-1.479$ or $x>2.479$

4 $3x+y=12$ ①

$4x-2y=26$ ②

Multiply ① by 2.

$6x+2y=24$ ③

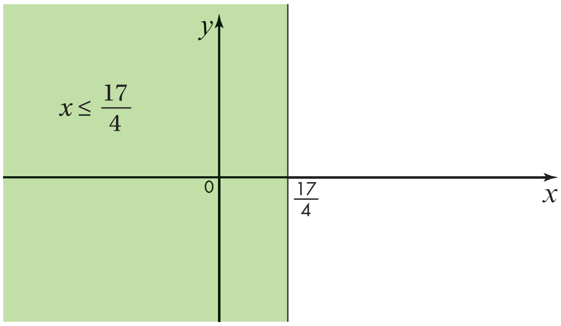
Add ② and ③.

$10x=50$

$x=5$ and $y=-3$

5 $4x-7\leqslant 10$

$x\leqslant\frac{17}{4}$



6 $(x-10)(x+1)=0$

$x=10$ or -1

So $x\geqslant 10$ or $x\leqslant -1$

7 x-intercepts at $(-\frac{1}{2}, 0)$ and (2, 0) and y-intercept at (2, 0); $a<0$

$y=(-2x-1)(x-2)$

$y=-2x^2+3x+2$

Satisfies the intercepts and $a<0$.

8 $y=2x^2+3x+13$

$b^2-4ac=9-(4)(2)(13)=-95<0$ so no real roots.

The statement '$2x^2+3x+13\geqslant 0$ for all values of x' is correct. The discriminant is < 0, consequently the equation has no real roots and so it does not intercept the x-axis. As $a>0$ all of the curve will be above the x-axis.

9 $y = 4x^2 - 7x - 2$

$y = 4\left(x^2 - \frac{7x}{4} - \frac{1}{2}\right)$

$y = 4\left[\left(x - \frac{7}{8}\right)^2 - \frac{49}{64} - \frac{1}{2}\right]$

$y = 4\left[\left(x - \frac{7}{8}\right)^2 - \frac{81}{64}\right]$

At the turning point y will have a minimum value i.e. when $x = \frac{7}{8}$.

When $x = \frac{7}{8}$,

$y = 4\left(-\frac{81}{64}\right)$

$y = -\frac{81}{16}$

So coordinates of turning point are $(\frac{7}{8}, -\frac{81}{16})$

10 $3x + 5y = 2$ ①

$2x - 6y = 6$ ②

Multiply ① by 2 and multiply ② by 3.

$6x + 10y = 4$ ③

$6x - 18y = 18$ ④

Subtract ④ from ③.

$28y = -14$

$y = -\frac{1}{2}$ and $x = \frac{3}{2}$

So the coordinates of the point of intersection of these two lines are $(\frac{3}{2}, -\frac{1}{2})$.

11 $3x - 2y = 7$ ①

$x^2 - y^2 = 8$ ②

Rearrange ①.

$x = \frac{7 + 2y}{3}$

Substitute into ②.

$\left(\frac{7 + 2y}{3}\right)^2 - y^2 = 8$

$49 + 28y + 4y^2 - 9y^2 = 72$

$5y^2 - 28y + 23 = 0$

$(5y - 23)(y - 1) = 0$

$(\frac{27}{5}, \frac{23}{5})$ and $(3, 1)$

12 a $6 - 9x \leqslant 4x - 5$

$13x \geqslant 11$

$x \geqslant \frac{11}{13}$

$2x > 35x + 20$

$x < -\frac{20}{33}$

b No solution (as no overlap of inequalities).

13 $6x^2 > 3 - 7x$

$6x^2 + 7x - 3 > 0$

$(2x + 3)(3x - 1) > 0$

$x < -\frac{3}{2}$ or $x > \frac{1}{3}$

14 a $x^2 + 10x + 21 > 0$

$(x + 3)(x + 7) > 0$

$x < -7$ or $x > -3$

and $3x - 5 > 0$

$x > \frac{5}{3}$

So $x > \frac{5}{3}$, i.e. {all integers greater than $\frac{5}{3}$}

b $x^2 + 10x + 21 > 0$ or $3x - 5 > 0$

$(x + 3)(x + 7) > 0$

$x < -7$ or $x > -3$

or $3x - 5 > 0$

$x > \frac{5}{3}$

So $x < -7$ or $x > -3$ {all integers less than –7 or greater than –3}

15 $3x^2 + 5x - 2 > 7x + 15$

$3x^2 - 2x - 17 > 0$

$x = \frac{2 \pm \sqrt{4 - (4)(3)(-17)}}{6}$

$x = \frac{2 \pm \sqrt{208}}{6}$

$x = -2.07$ or $x = 2.737$

$x < -2.07$ or $x > 2.737$

16 $ax^2 + 5x - 6 = 0$

a $b^2 - 4ac < 0$

$25 - (4)(a)(-6) < 0$

$25 + 24a < 0$

$24a < -25$

$a < -\frac{25}{24}$

b $b^2 - 4ac = 0$

$25 - (4)(a)(-6) = 0$

$25 + 24a = 0$

$25 = -24a$

$a = -\frac{25}{24}$

c $b^2 - 4ac > 0$

$25 - (4)(a)(-6) > 0$

$25 + 24a > 0$

$24a > 25$

$a > -\frac{25}{24}$

17 $a^2x^2+3bx+\sqrt{c}=0$

$$a^2\left(x^2+\frac{3bx}{a^2}+\frac{\sqrt{c}}{a^2}\right)=0$$

$$\left(x+\frac{3b}{2a^2}\right)^2-\frac{9b^2}{4a^4}+\frac{\sqrt{c}}{a^2}=0$$

$$\left(x+\frac{3b}{2a^2}\right)^2=\frac{9b^2}{4a^4}-\frac{\sqrt{c}}{a^2}$$

$$\left(x+\frac{3b}{2a^2}\right)^2=\frac{9b^2}{4a^4}-\frac{4a^2\sqrt{c}}{4a^4}$$

$$\left(x+\frac{3b}{2a^2}\right)^2=\frac{9b^2-4a^2\sqrt{c}}{4a^4}$$

$$x+\frac{3b}{2a^2}=\frac{\pm\sqrt{9b^2-4a^2\sqrt{c}}}{2a^2}$$

$$x=\frac{-3b\pm\sqrt{9b^2-4a^2\sqrt{c}}}{2a^2}$$

18 Perimeter:

$2(x+y)+2(2x+3y)=18$

$6x+8y=18$

$3x+4y=9$ ①

$(x+y)(2x+3y)=14$

$2x^2+5xy+3y^2-14=0$ ②

Rearrange ①.

$$y=\frac{9-3x}{4}$$

Substitute into ②.

$$2x^2+5x\left(\frac{9-3x}{4}\right)+3\left(\frac{9-3x}{4}\right)^2-14=0 \quad ③$$

$x^2-18x-19=0$

$(x-19)(x+1)=0$

$x=19$ or -1

$y=-12$ or 3

So width $=19+-12=7$ or $-1+3=2$

So length $=38-36=2$ or $-2+9=7$

19 $13\pi r\leqslant 260$

$r\leqslant 6.4$

$\pi r^2>60$

$r>4.4$

$4.4<r\leqslant 6.4$

20 $2x-3y=-5$ ①

$3y+7z=2$ ②

$7xz=-2$ ③

From ③ $z=-\dfrac{2}{7x}$

Substitute into ② and rearrange.

$$3y-\frac{2}{x}=2 \quad ④$$

From ① $x=\dfrac{3y-5}{2}$

Substitute into ④ and solve.

$$3y-\frac{4}{3y-5}=2$$

$3y^2-7y+2=0$

$(3y-1)(y-2)=0$

$y=\dfrac{1}{3}$ or 2

$x=-2$ or $\dfrac{1}{2}$

$z=\dfrac{1}{7}$ or $-\dfrac{4}{7}$

One set of solutions is $y=\dfrac{1}{3}, x=-2, z=\dfrac{1}{7}$

Another set of solutions is $y=2, x=\dfrac{1}{2}, z=-\dfrac{4}{7}$

3 Algebra and functions 3: Sketching curves

Prior knowledge p 59

1 **a** $(x-7)(x+4)=0$ so $x=-4$ or 7

b $(x+7)(x-7)=0$ so $x=-7$ or 7

c $(x-3)^2=0$ so $x=3$

d $5x^2-x=0$

$x(5x-1)=0$ so $x=0$ or $\frac{1}{5}$

e $(3x-5)(3x+5)=0$ so $x=-\frac{5}{3}$ or $\frac{5}{3}$

f $(3x-5)(2x-1)=0$ so $x=\frac{5}{3}$ or $\frac{1}{2}$

2 **a** $(x-4)^2-17=0$

$x-4=\pm\sqrt{17}$

$x=4\pm\sqrt{17}$

b $(x-1)^2-4=0$

$x-1=\pm\sqrt{4}$

$x=1\pm\sqrt{4}$

$x=-1$ or 3

c $(x+3)^2-7=0$

$x+3=\pm\sqrt{7}$

$x=-3\pm\sqrt{7}$

3 **a** $(4, -17)$, minimum as $a>0$

b $(1, -4)$, minimum as $a>0$

c $(-3, -7)$, minimum as $a>0$

4 **a** $a=1$, $b=-6$, $c=9$

$b^2-4ac=36-(4\times1\times9)=0$ so one real root.

b $a=1$, $b=-2$, $c=15$

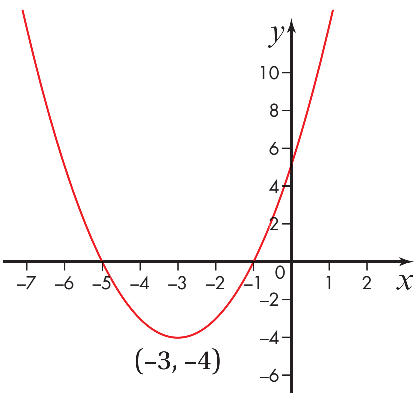
$b^2-4ac=4-(4\times1\times15)=-56$ so no real roots.

c $a=1$, $b=-7$, $c=-1$

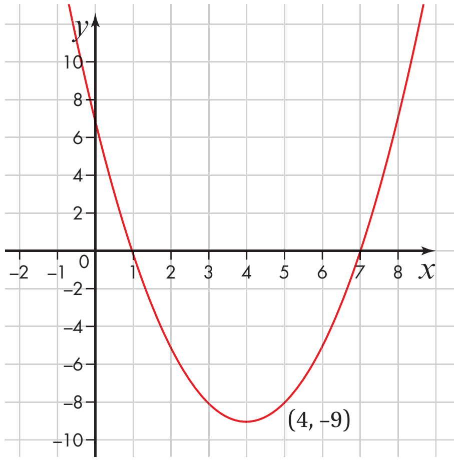
$b^2-4ac=49-(4\times1\times-1)=53$ so two real roots.

Exercise 3.1A p 62

1 **a**



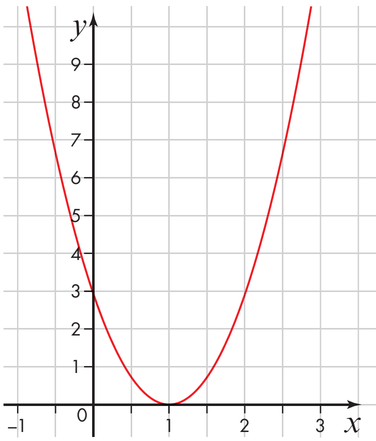
b



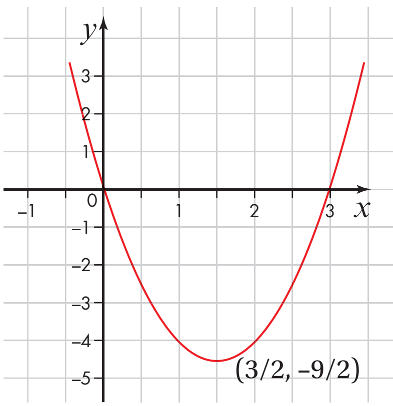
2 Roots at $x=1$ and $x=3$ so factors are $(x-1)$ and $(x-3)$.

$y=(x-1)(x-3)=x^2-4x+3$

3 **a**



b



4 The sketch is *not* correct.

$x^2-4x-21=(x-7)(x+3)$

So should cut x-axis at $(-3, 0)$ and $(7, 0)$.

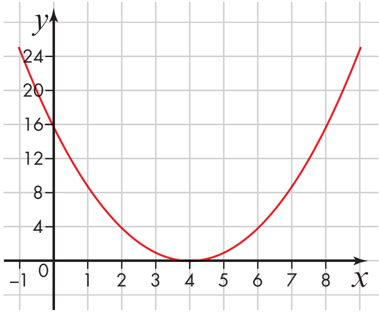
$a>0$ so graph should be U-shaped.

When $x=0$, $y=-21$ so y-intercept at $(0, -21)$.

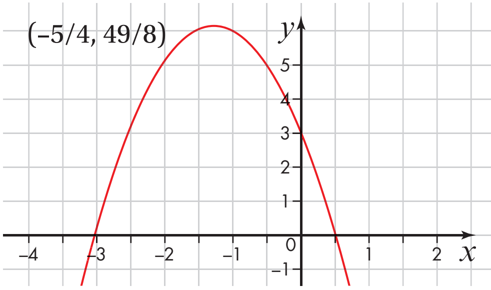
$x^2-4x-21=(x-2)^2-25$

So turning point at $(2, -25)$.

5 a



b



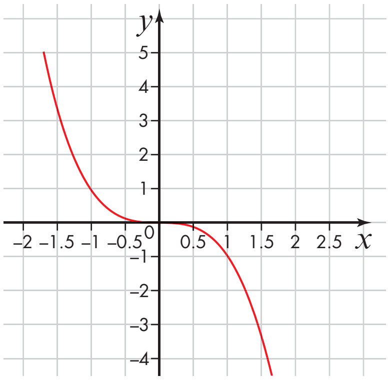
6 a 4 m **b** 3 s

c Roots at $x = -1$ and $x = 3$ so factors are $(x + 1)$ and $(x - 3)$, but graph shape indicates a < 0, so $-(x + 1)(x - 3) = 3 + 2x - x^2$

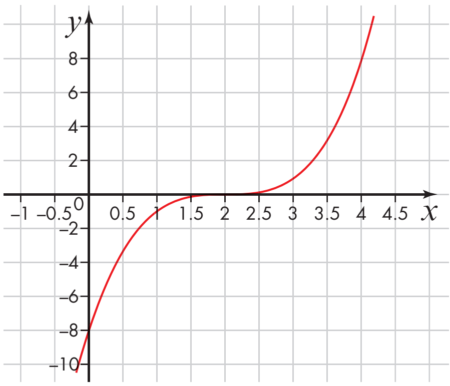
$y = 3 + 2x - x^2$

Exercise 3.2A p 64

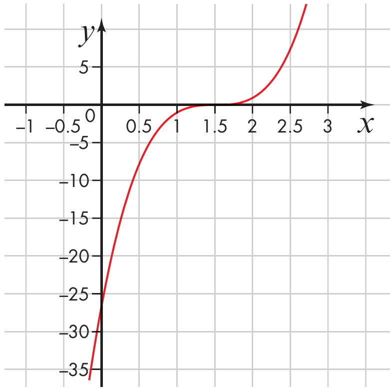
1 a



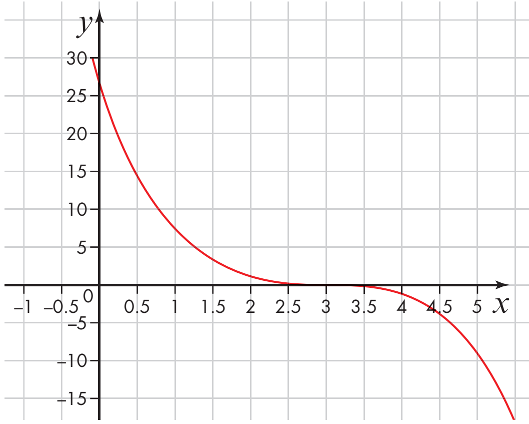
b



2 a



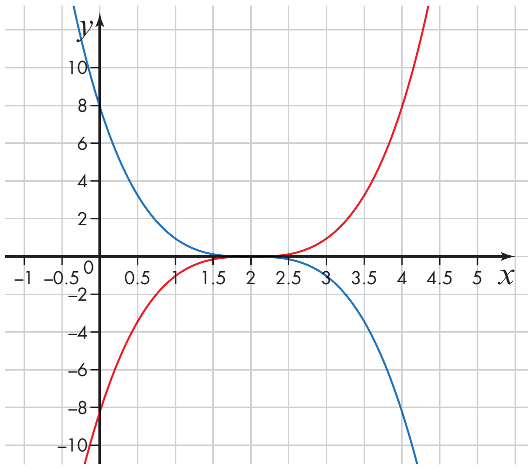
b



3 The sketch is *not* correct.

Point of inflection should be at (–3, 0) and curve should be a reflection in the y-axis of the curve shown.

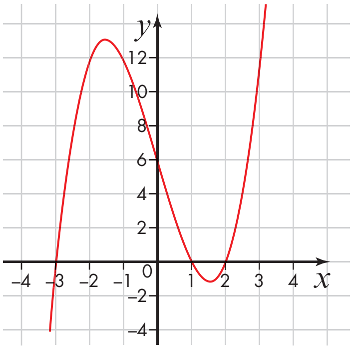
4 The graphs are reflections of each other in the line $x = 2$.



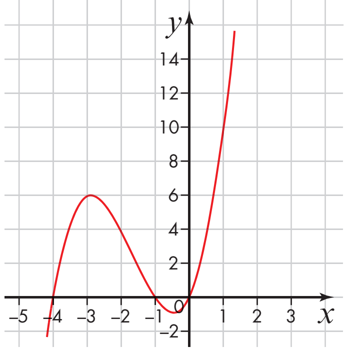
5 C $y = (3 - 4x)^3$ only

Exercise 3.2B p 66

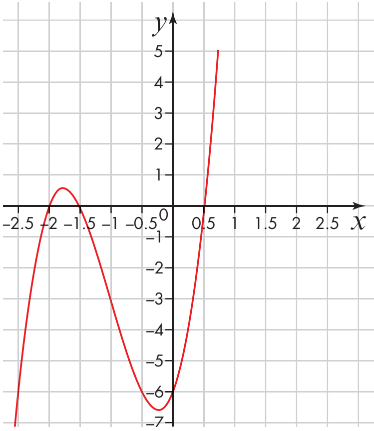
1 a



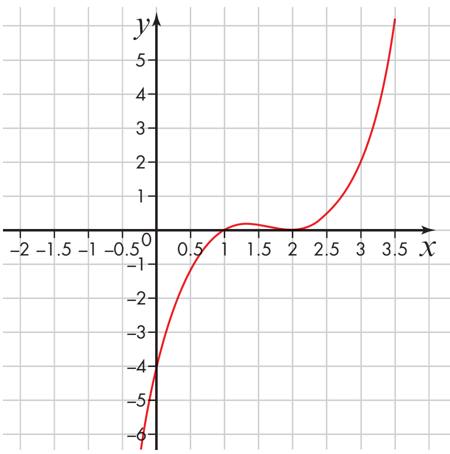
b



2 a

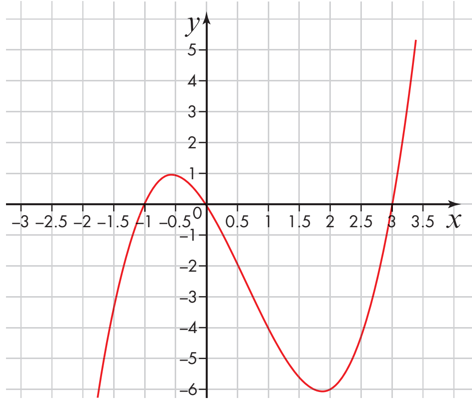


b

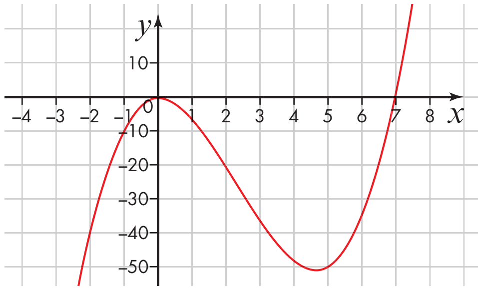


3 a $x^3 - 2x^2 - 3x = x(x^2 - 2x - 3)$

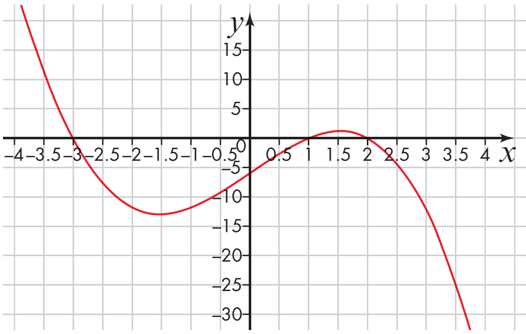
$= x(x - 3)(x + 1)$



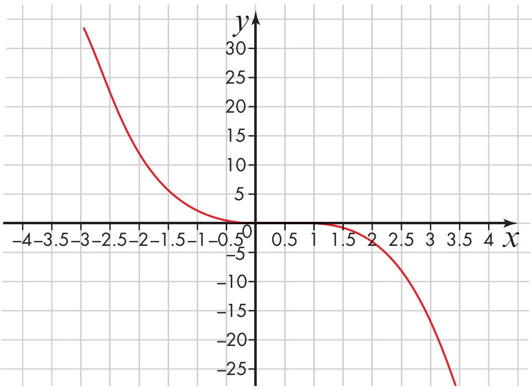
b $x^3 - 7x^2 = x^2(x - 7)$



4 a



b

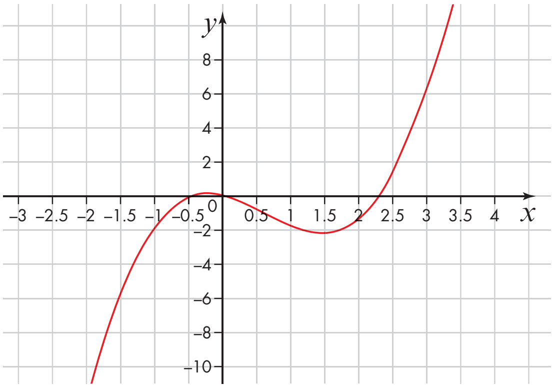


5 $x^3 - 2x^2 - x = x(x^2 - 2x - 1)$

Using the quadratic formula to solve

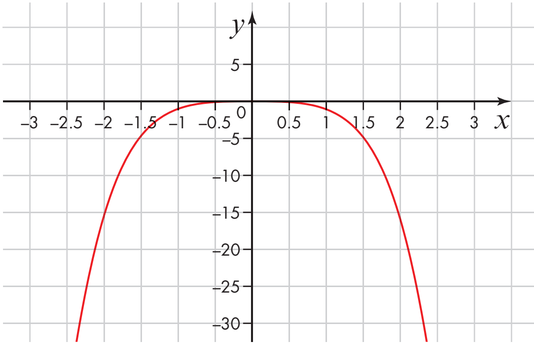
$x^2 - 2x - 1 = 0$

$1 - \sqrt{2} = -0.41$ or $1 + \sqrt{2} = 2.41$.

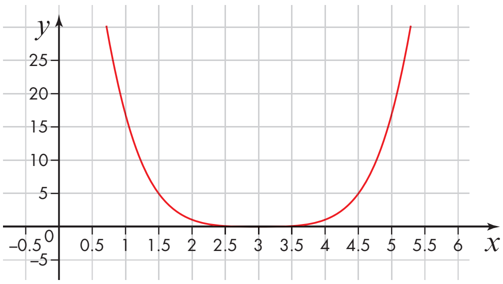


Exercise 3.3A p 69

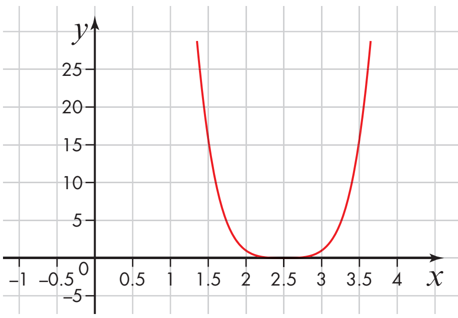
1 a



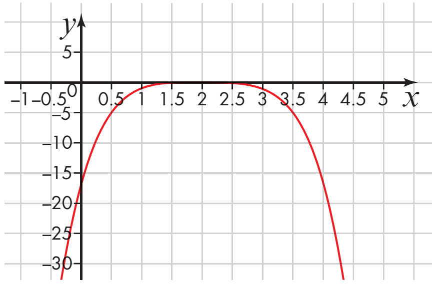
b



2 a



b



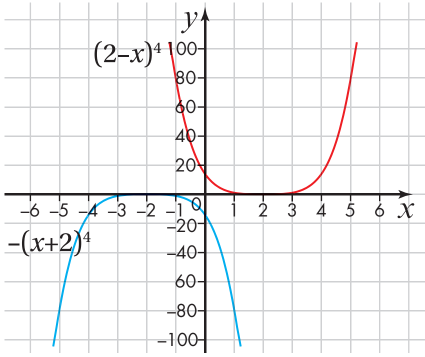
3 The curve is the wrong shape: it should *not* be a parabola.

The curve should be a U-shape sitting on the x-axis at (3, 0) and intercepting the y-axis at (0, 81).

4 They do not generate the same curve because they are different functions.

$-(x + 2)^4 = -(x + 2)(x + 2)(x + 2)(x + 2)$

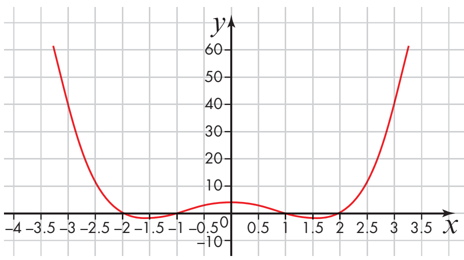
$(2 - x)^4 = (2 - x)(2 - x)(2 - x)(2 - x)$



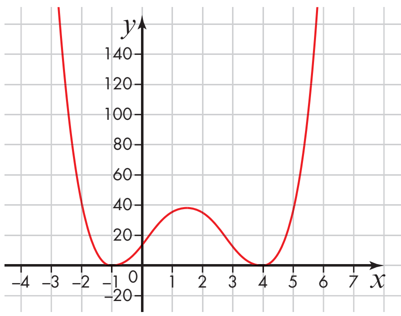
5 **B** $y = (4 - 5x)^4$ and **C** $y = (5x - 4)^4$

Exercise 3.3B p 72

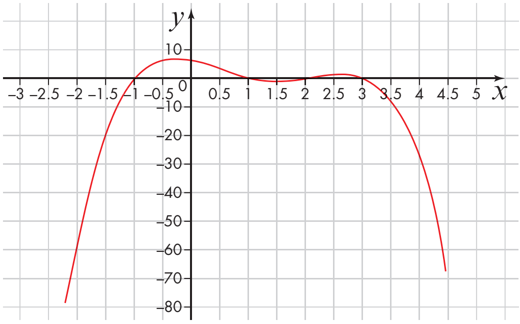
1 a



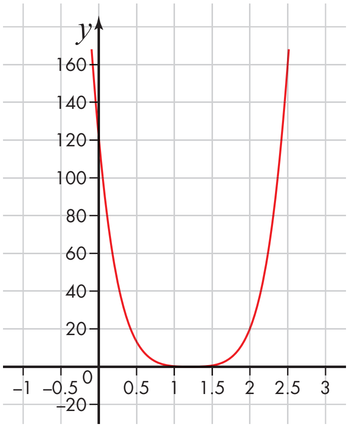
b



2 **a**

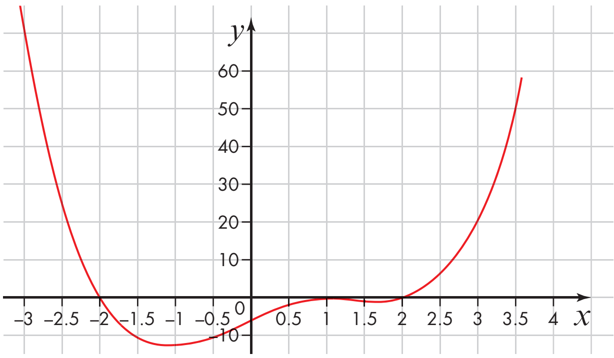


b

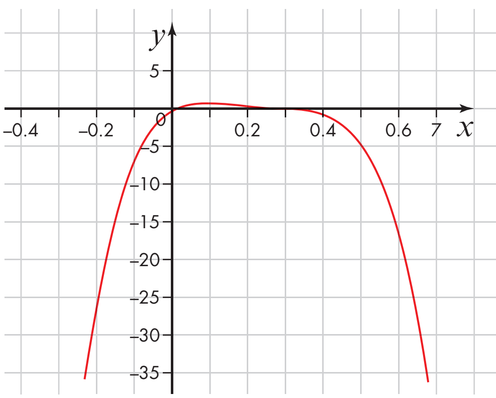


3 $x(x-1)(x+2)(x+3)$

4 **a**



b

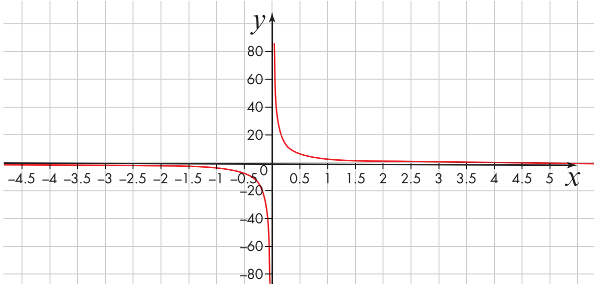


5 $(x-3)^2(x+3)^2$

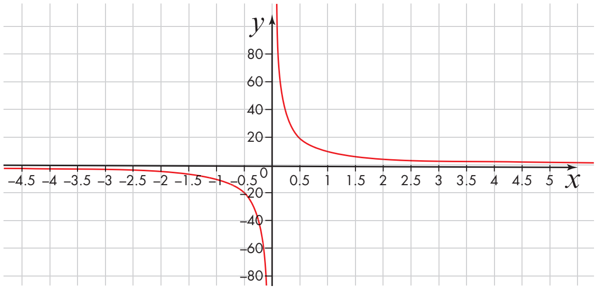
Exercise 3.4A

p 74

1 **a** Asymptotes at $x = 0$ and $y = 0$.



b Asymptotes at $x = 0$ and $y = 0$.



c In both graphs the asymptotes are in the same locations. The key difference is that the curves of the graph of $y = \frac{5}{x}$ are much closer to the origin than those of $y = \frac{10}{x}$.

2 **a** y-intercept at (0, 1); asymptotes at $x = -2$ and $y = 0$.

b y-intercept at (0, 0.75); asymptotes at $x = 4$ and $y = 0$.

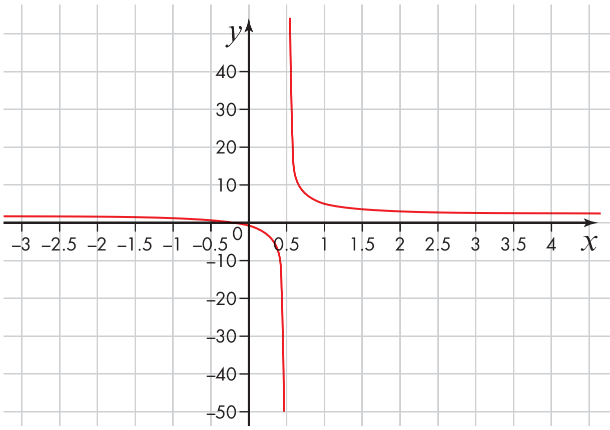
3 **a** $y = \frac{1}{(x+1)}$

b $y = \frac{-2}{(x+1)}$

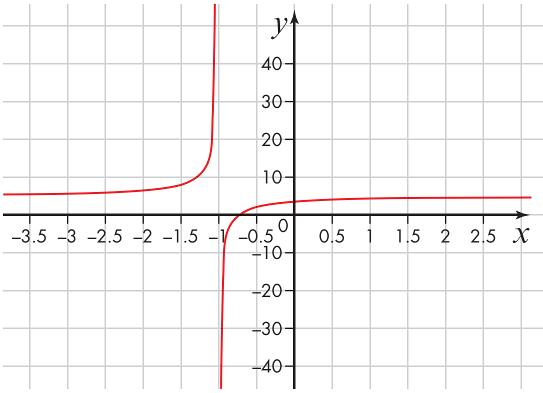
4 **a** iv **b** ii

c iv **d** i

5 **a** Asymptotes at $x = \frac{1}{2}$ and $y = 3$; y-intercept at (0, –1).



b Asymptotes at $x = -1$ and $y = 5$; y-intercept at (0, 3).

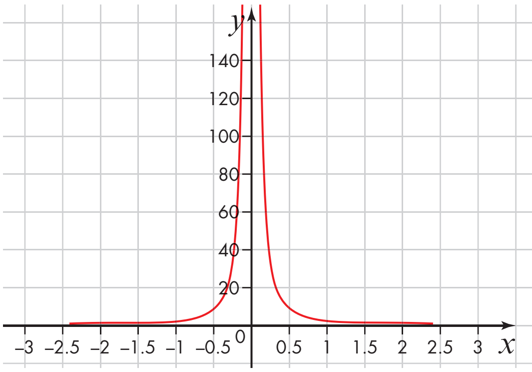


6 No. The asymptotes are in the correct locations at $x = \frac{2}{3}$ and $y = 2$. However, the graph drawn is for a reciprocal with a *positive* numerator whereas the given equation has a negative numerator.

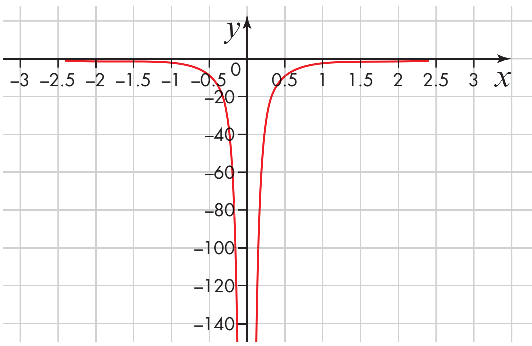
Exercise 3.4B

p 78

1 a Asymptotes at $x = 0$ and $y = 0$.

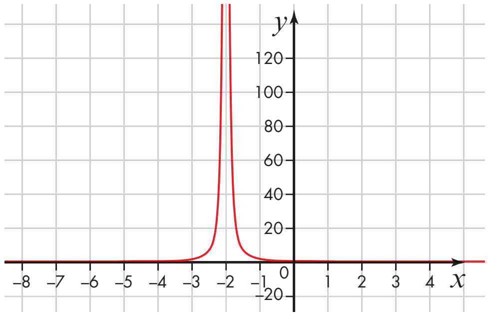


b Asymptotes at $x = 0$ and $y = 0$.

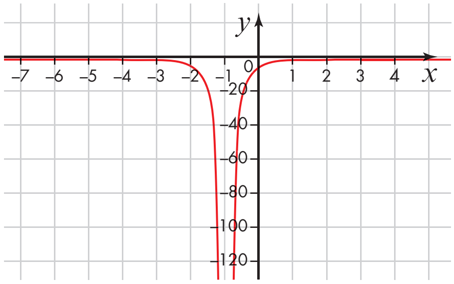


c In both graphs the asymptotes are in the same locations. The key differences are (**i**) that the curves of the graph of $y = \frac{3}{x^2}$ are much closer to the origin than those of $y = \frac{-6}{x^2}$ and (**ii**) the graph of $y = \frac{-6}{x^2}$ is a reflection in the x-axis.

2 a Asymptotes at $x = -2$ and $y = 0$; y–intercept at (0, 0.25).



b Asymptotes at $x = -1$ and $y = 0$; y-intercept at (0, −5).

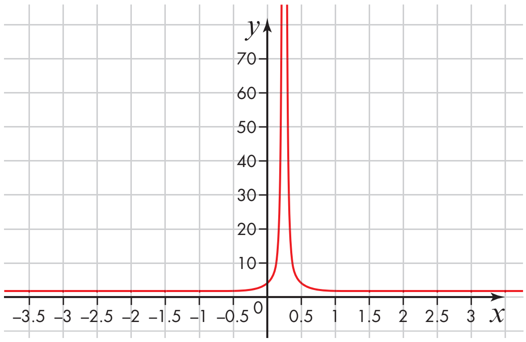


3 a $y = \dfrac{1}{(x-3)^2} + 2$ **b** $y = \dfrac{1}{(x-4)^2} - 3$

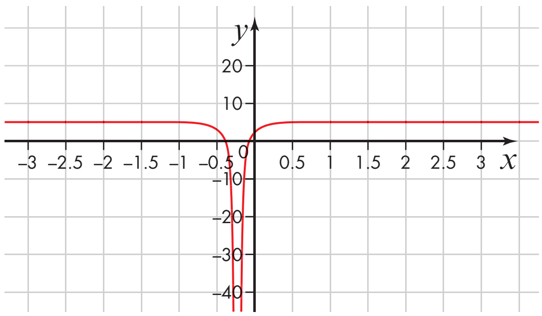
4 a iii **b** i

c ii **d** i

5 a Asymptotes at $x = \frac{1}{4}$ and $y = 2$; y-intercept at (0, 5).



b Asymptotes at $x = -\frac{1}{5}$ and $y = 6$; y-intercept at (0, 3).

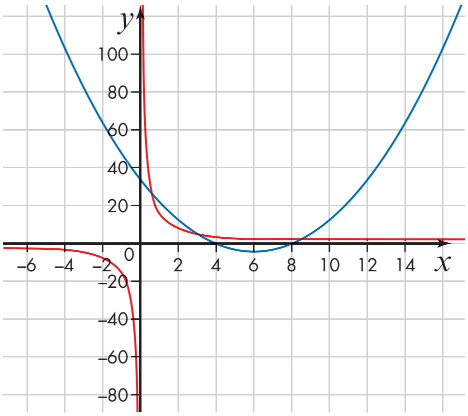


6 Yes, the graph is for the given function. Correct shape with asymptotes at $x = \frac{3}{2}$ and $y = 2$; y-intercept at $(0,\ 2\frac{7}{9})$.

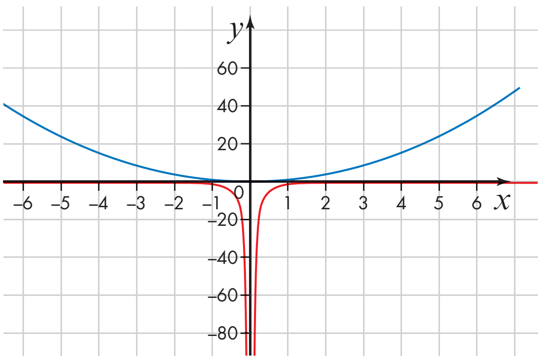
Exercise 3.5A

p 82

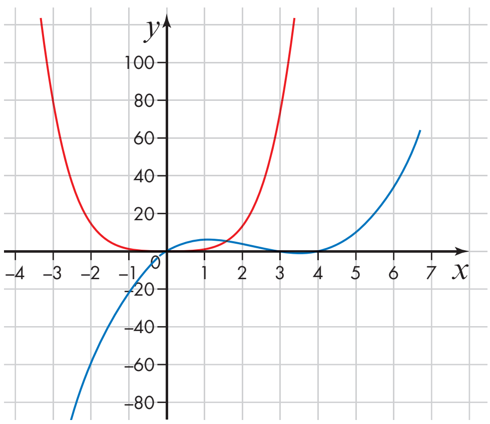
1 There are three points of intersection.



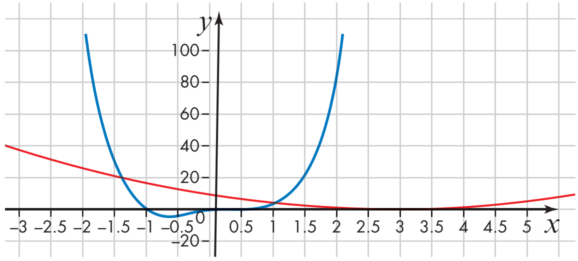
2 There are no solutions to $\frac{1}{x^2} = -x^2$ because the graphs do not intersect.



3 There are two solutions to $x^4 = x(x-3)(x-4)$ because the graphs intersect twice.



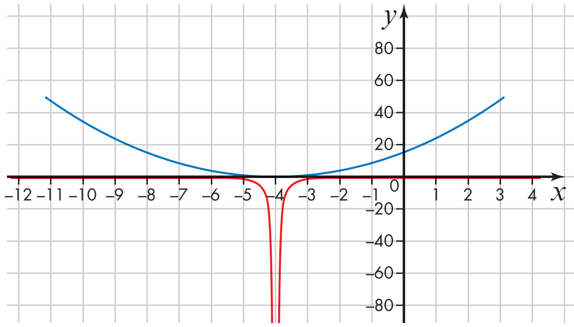
4 $\frac{(x+1)(2x-1)^3}{(3-x)^2} = 1$ rearranged is $(x+1)(2x-1)^3 = (3-x)^2$. There are two solutions to $\frac{(x+1)(2x-1)^3}{(3-x)^2} = 1$ because the graphs intersect twice.



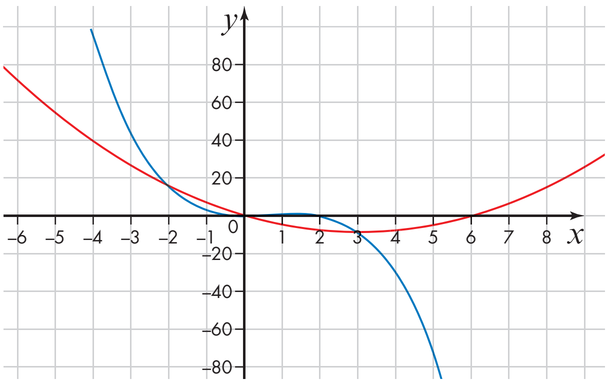
5 $(x+4)^4 = -2$ rearranged is $-\frac{2}{(x+4)^2} = (x+4)^2$.

There are no solutions to $(x+4)^4 = -2$ because the graphs do not intersect.

Also, there are no real solutions to the fourth root of a variable with a value of –2.



6 At points of intersection, the equations are equal.

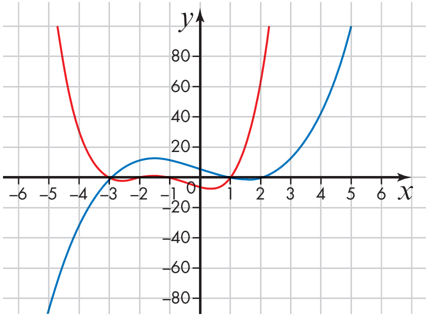


$$x(x-6) = x^2(2-x)$$
$$x^2 - 6x = 2x^2 - x^3$$
$$x^3 - x^2 - 6x = 0$$
$$x(x-3)(x+2) = 0$$
$$x = -2,\ 0 \text{ or } 3$$

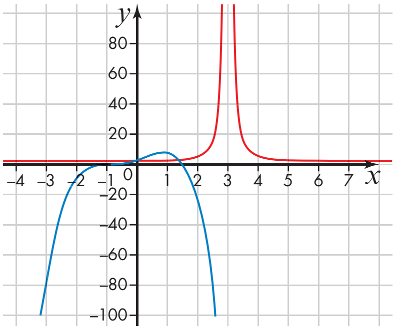
Substituting into one of the original equations:

$$y = 16,\ 0 \text{ or } -9$$

7 There are two points of intersection.



8 There are two points of intersection.



Exercise 3.6A

p 85

1 a $y \propto x^3$

$y = kx^3$

$k = \frac{16}{8} = 2$

$y = 2x^3$

When $x = 3$, $y = 2(3)^3 = 54$

b $250 = 2x^3$

$x = 5$

2 $d \propto t$

$d = kt$

$k = \frac{15}{0.25} = 60$

Constant speed is 60 mph.

Using the graph, when $t = 4.5$, $d = 270$.

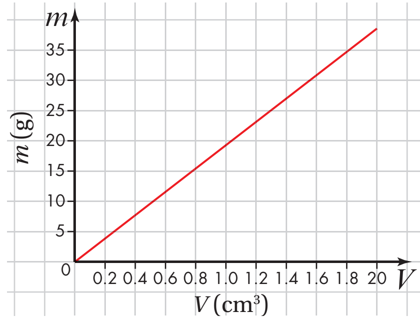
The distance travelled is 270 miles.

3 $m \propto V$

$m = kV$

$k = \frac{9.66}{0.5} = 19.32$

When $V = 0.75$, $m = 19.32 \times 0.75 = 14.49$ g



Using the graph, when $m = 15$, $V = 0.78$.

The volume of the ring is $0.78\ \text{cm}^{-3}$.

4 a C, iii **b** A, ii **c** B, i

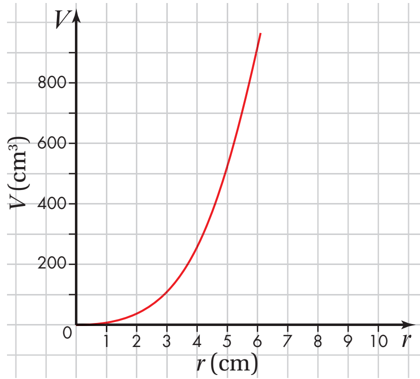
5 $V \propto r^3$

$V = kr^3$

$k = \frac{36\pi}{27} = \frac{4\pi}{3}$

$V = \frac{4\pi}{3}r^3$

When $r = 7$, $V = \frac{4\pi}{3}(7)^3 = \frac{1372\pi}{3}$



a Using the graph, when $r = 4$, $V = 268\ \text{cm}^3$.

b Using the graph, when $V = 288\pi$, $r = 6$ cm.

c Substituting $r = 4$ into the equation gives $V = 268$.

Likewise substituting $V = 288\pi$ into the equation gives $r = 6$.

6 a $y \propto \sqrt{x}$

$y = k\sqrt{x}$

From the graph, when $x = 4$, $y = 10$.

$10 = k\sqrt{4}$

$k = 5$

$y = 5\sqrt{x}$

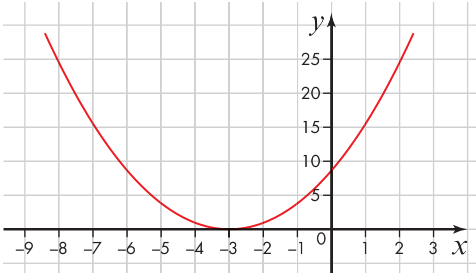
b $y = 5\sqrt{16} = 20$

c $50 = 5\sqrt{x}$

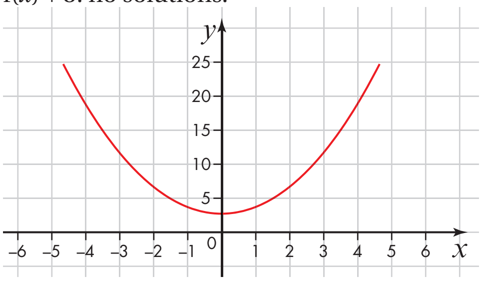
$x = 100$

Exercise 3.7A p 89

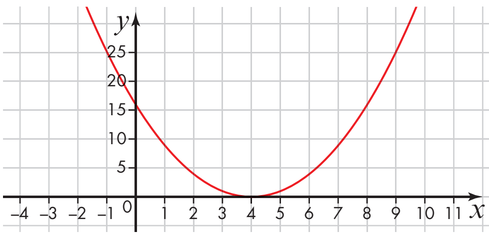
1 **a** $f(x+3)$: one solution.



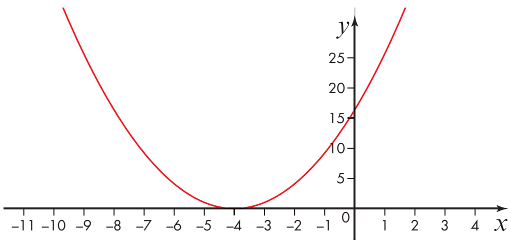
b $f(x)+3$: no solutions.



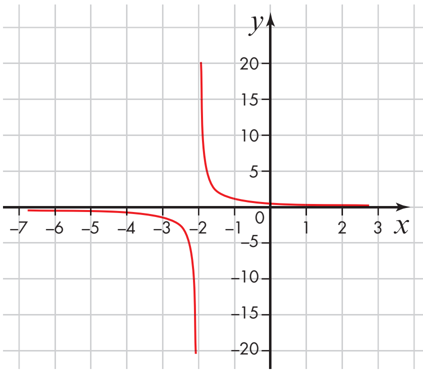
c $f(x-4)$: one solution.



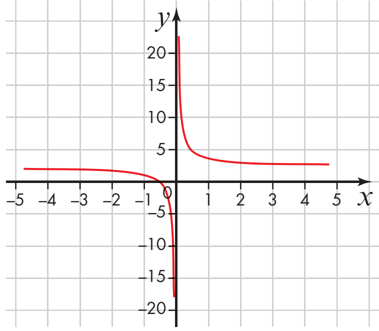
d $f(x+4)$: one solution.



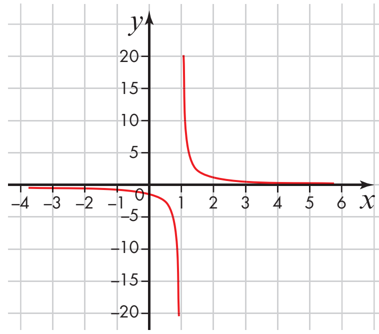
2 **a** $f(x+2)$: asymptotes at $x=-2$ and $y=0$.



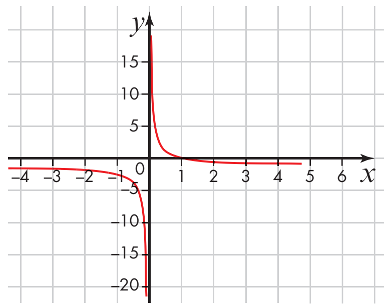
b $f(x)+2$: asymptotes at $x=0$ and $y=2$.



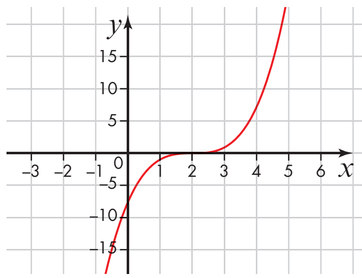
c $f(x-1)$: asymptotes at $x=1$ and $y=0$.



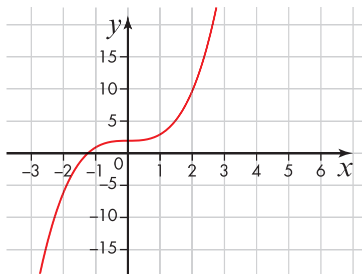
d $f(x)-1$: asymptotes at $x=0$ and $y=-1$.



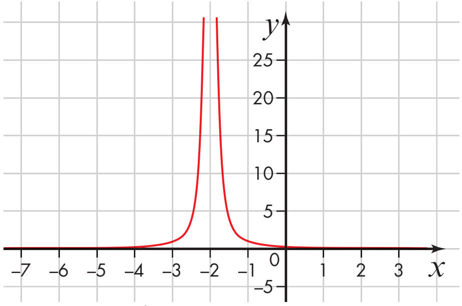
3 $f(x-2)$: Q at (2, 0); one solution.



$f(x)+2$: Q at (0, 2); one solution.



4 **a** $f(x+2)$: asymptotes at $x = -2$ and $y = 0$.



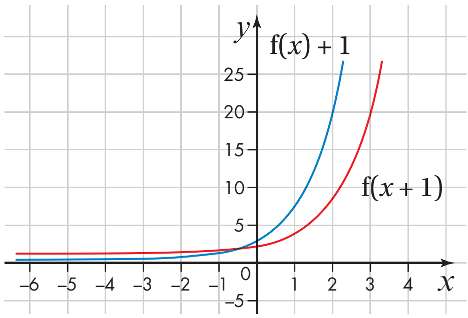
b $f(x+2) = \frac{1}{(x+2)^2}$

c $(0, \frac{1}{4})$

5 $f(x+1)$: P at $(-1, 1)$; asymptote at $y = 0$.

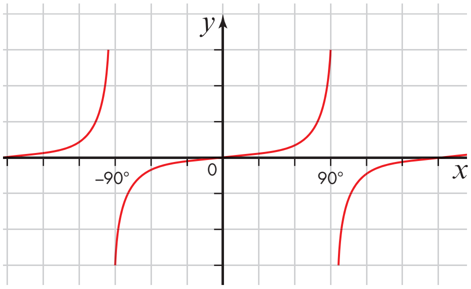
$f(x) + 1$: P at $(0, 2)$; asymptote at $y = 1$.

$f(x+1)$: $= f(x) + 1$ has one solution (one point of intersection).

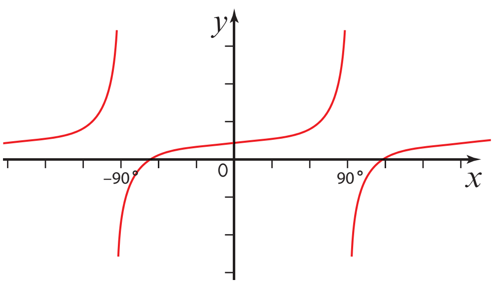


6 **a, b** $f(x - 180)$ where $-180° \leqslant x \leqslant 180°$: asymptotes at $x = -90$ and 90.

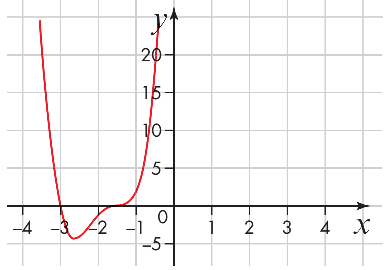
Axis intercepts at $(-180, 0)$, $(0, 0)$ and $(180, 0)$.



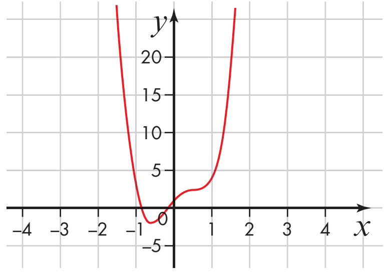
c, d $f(x) + 2$: asymptotes at $x = -90$ and 90.



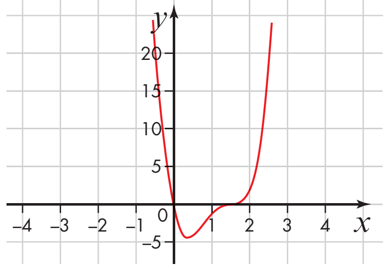
7 **a** $f(x+2) = (x+3)(2x+3)^3$: two solutions.



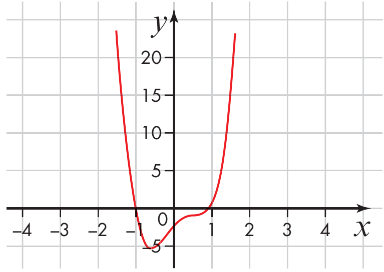
b $f(x) + 2 = (x+1)(2x-1)^3 + 2$: two solutions.



c $f(x-1) = x(2x-3)^3$: two solutions.

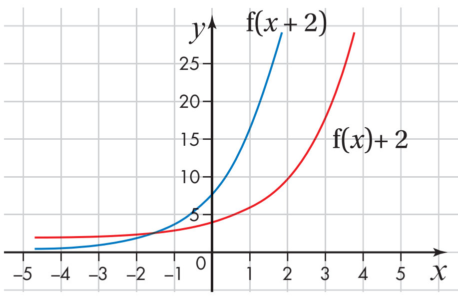


d $f(x) - 1 = (x+1)(2x-1)^3 - 1$: two solutions.



8 **a, b** $f(x+2) = 2^{x+3}$: no solutions.

$f(x) + 2 = 2^{x+1} + 2$: no solutions.



c If $f(x) = 2^x$ then $f(x+3) = 2^{x+3}$
and $f(x+1) + 2 = 2^{x+1} + 2$

9 $f(x) = x^4 - 5x^3 + 5x^2 + 5x - 6$

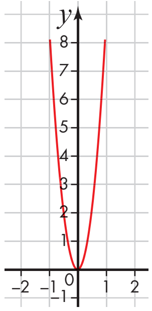
$f(x-1) = (x-1)^4 - 5(x-1)^3 + 5(x-1)^2 + 5(x-1) - 6$

$= x^4 - 2x^3 + x^2 - 2x^3 + 4x^2 - 2x + x^2 - 2x$
$\quad + 1 - 5x^3 - 5x + 15x^2 - 10x + 5$
$\quad + 5x^2 - 10x + 5 + 5x - 5 - 6$

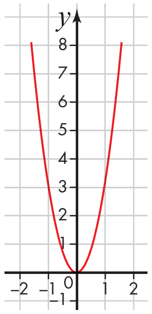
$= x^4 - 9x^3 + 26x^2 - 24x$

Exercise 3.8A p 95

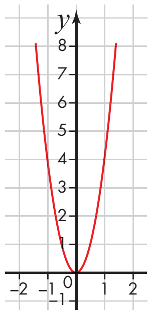
1 a $f(3x)$: one solution.



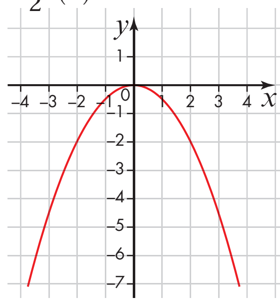
b $3f(x)$: one solution.



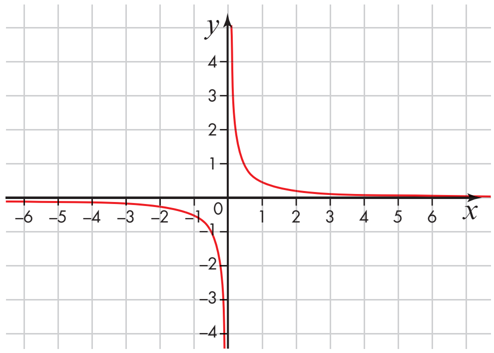
c $f(2x)$: one solution.



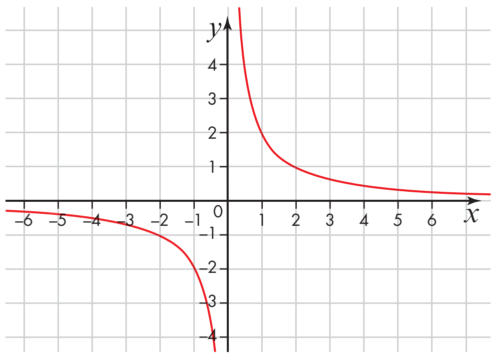
d $-\frac{1}{2}f(x)$: one solution.



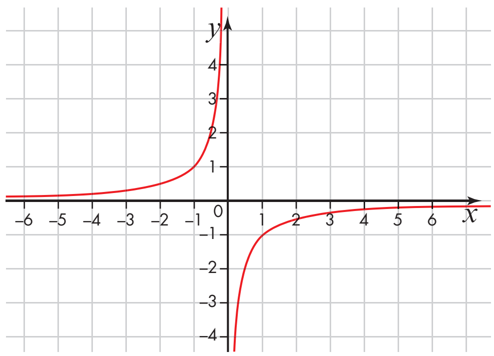
2 a $f(2x)$: asymptotes at $x = 0$ and $y = 0$.



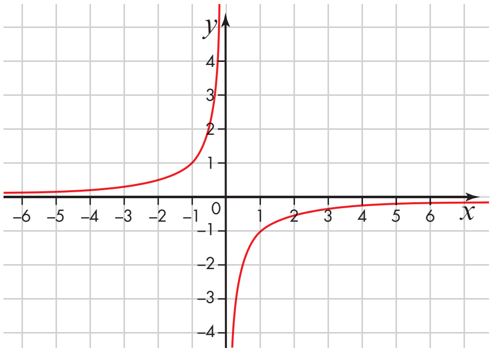
b $2f(x)$: asymptotes at $x = 0$ and $y = 0$.



c $f(-x)$: asymptotes at $x = 0$ and $y = 0$.

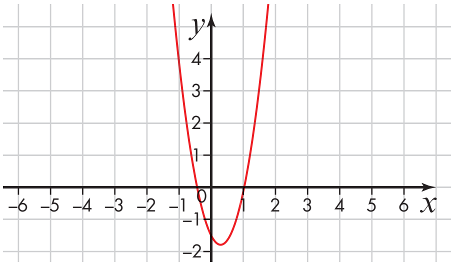


d $-\mathrm{f}(x)$: asymptotes at $x = 0$ and $y = 0$.



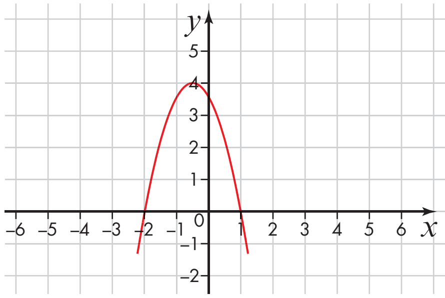
3 $\mathrm{f}(-2x)$: P(1, 0), R($-\frac{1}{2}$, 0), Q(0, –2).

$\mathrm{f}(-2x) = 0$ has two solutions.

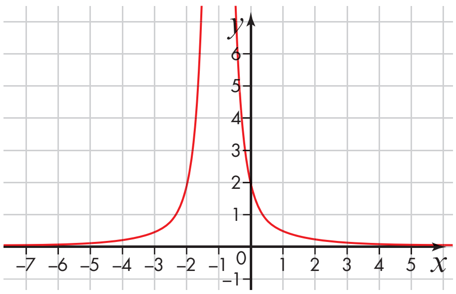


$-2\mathrm{f}(x)$: P(–2, 0), R(1, 0), Q(0, 4).

$-2\mathrm{f}(x) = 0$ has two solutions.



4 a $2\mathrm{f}(x)$: asymptotes at $x = -1$ and $y = 0$.



b $2\mathrm{f}(x) = \dfrac{2}{(x+1)^2}$

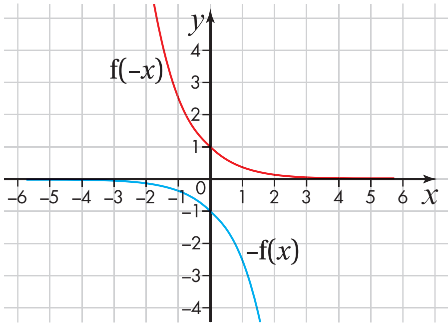
c (0, 2)

5 $\mathrm{f}(-x)$ and $-\mathrm{f}(x)$: in both cases asymptote at $y = 0$.

For $\mathrm{f}(-x)$, P is at (0, 1).

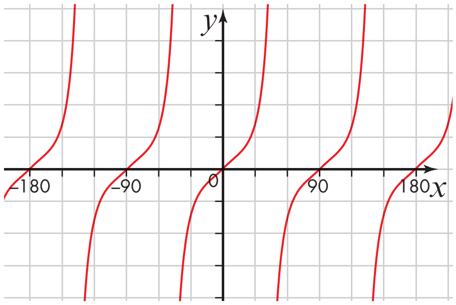
For $-\mathrm{f}(x)$, P is at (0, –1).

$\mathrm{f}(-x) = -\mathrm{f}(x)$: has no solutions (they do not intersect).



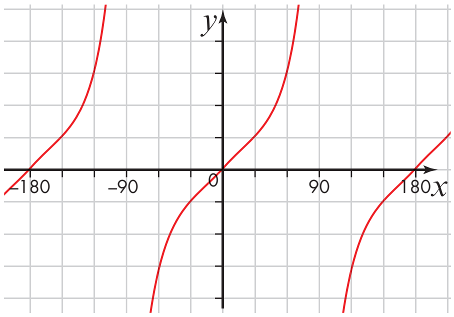
6 a, b $\mathrm{f}(2x)$ where $-180° \leqslant x \leqslant 180°$: asymptotes at $x = -135°, -45°, 45°$ and $135°$.

Axis intercepts at (–180°, 0), (–90°, 0), (0, 0), (90°, 0) and (180°, 0).

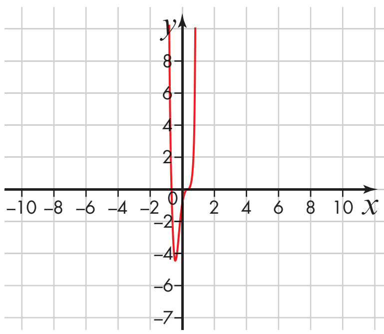


c, d $2\mathrm{f}(x)$ where $-180° \leqslant x \leqslant 180°$: asymptotes at $x = -90°$ and $90°$.

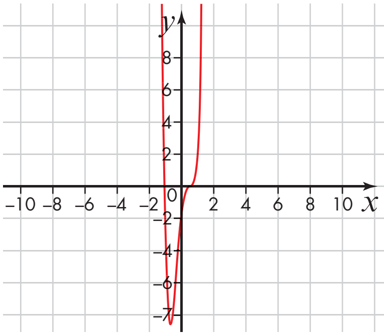
Axis intercepts at (–180°, 0), (0, 0) and (180°, 0).



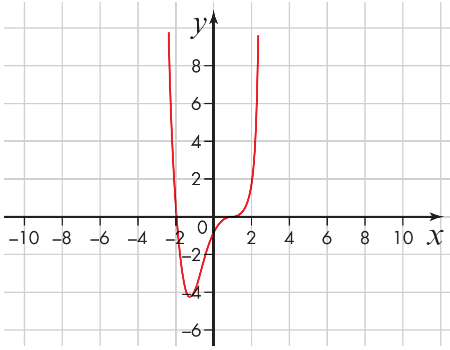
7 a $\mathrm{f}(2x) = (2x+1)(4x-1)^3$: two solutions.



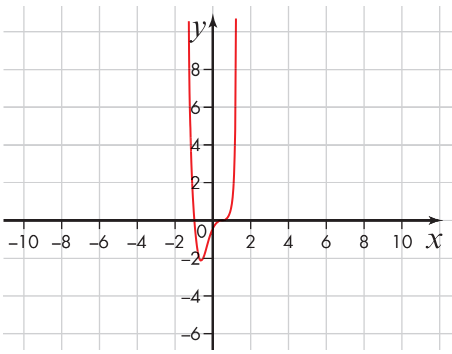
b $2f(x) = 2(x+1)(2x-1)^3$: two solutions.



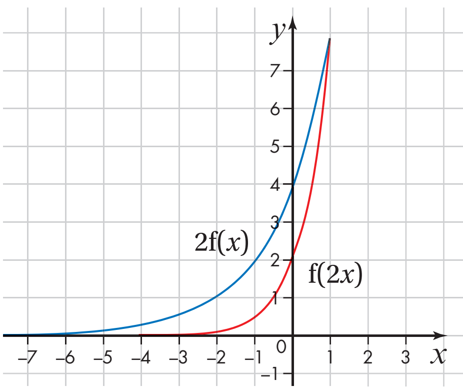
c $f\left(\frac{x}{2}\right) = \left(\frac{x}{2}+1\right)(x-1)^3$: two solutions.



d $\frac{1}{2}f(x) = \frac{1}{2}(x+1)(2x-1)^3$: two solutions.



8 **a, b, d** $f(2x) = 2^{2x+1}$: no solutions. $2f(x) = (2)(2^{x+1})$: no solutions.



c If $f(x) = 2^x$, $f(2x+1) = 2^{2x+1}$ and $2f(x+1) = (2)(2^{x+1})$

9 $f(x) = x^4 - 5x^3 + 5x^2 + 5x - 6$

$$f\left(\frac{x}{3}\right) = \left(\frac{x}{3}\right)^4 - 5\left(\frac{x}{3}\right)^3 + 5\left(\frac{x}{3}\right)^2 + 5\left(\frac{x}{3}\right) - 6$$

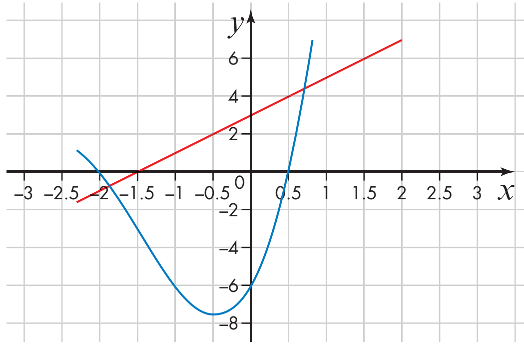
$$= \frac{x^4}{81} - \frac{5x^3}{27} + \frac{5x^2}{9} + \frac{5x}{3} - 6$$

Exam-style questions 3 p 98

1 **a** $f(x+2)$ **b** $f(-x)$

c $f(x)+1$ **d** $f(x)$

2 The cable will touch the rollercoaster at two points in the given interval (two points of intersection).



3 $r \propto p$

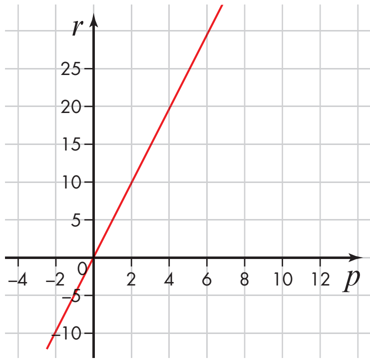
$r = kp$

$80 = 16k$

$k = 5$

$r = 5p$

The exchange rate is 5 reals to £1.



Using the graph:

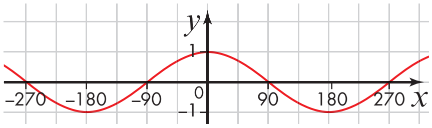
a $r = 5 \times 29$

$r = 145$

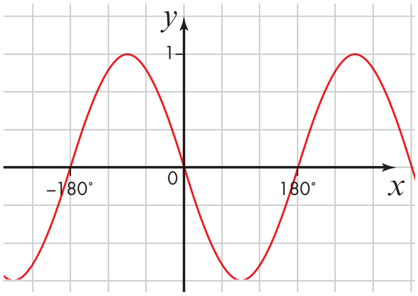
b $250 = 5 \times p$

$p = 50$

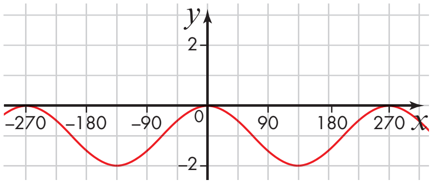
4 a f$(x + 90)$



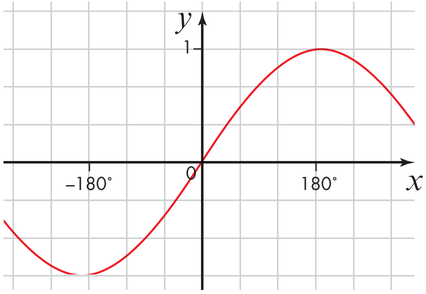
b $-$f(x)



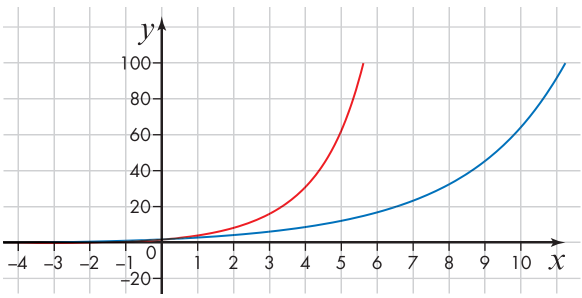
c f$(x) - 1$



d f$\left(\frac{x}{2}\right)$



5 $y = 2^{\frac{x}{2}+1}$ and $y = 2^{x+1}$



The rabbit and mice populations are only the same when they are both first introduced to the island.

6 $A \propto l^2$

$A = kl^2$

$254.504 = 81k$

$k = \pi$

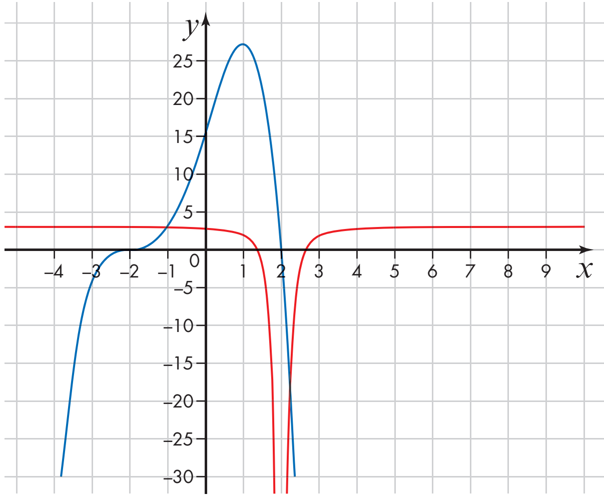
$A = \pi l^2$

When $l = 7$, $A = 49\pi$

"The shape is a circle"

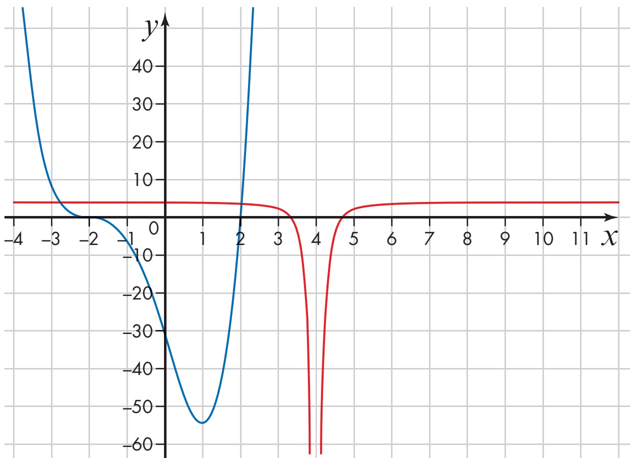
7 a Asymptotes at $x = 2$ and $y = 3$.

There are two points of intersection.



b Asymptotes at $x = 4$ and $y = 3$.

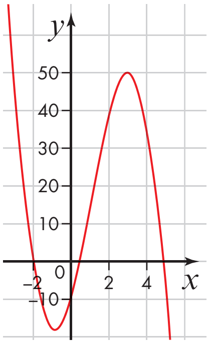
There are two points of intersection.



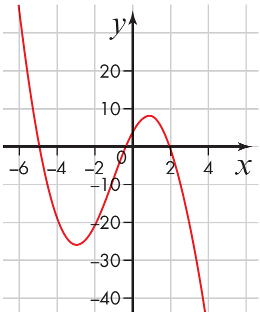
8 a f$(x) + 2$ **b** $-$f(x)

c f$(x - 2)$ **d** $f(-x)$

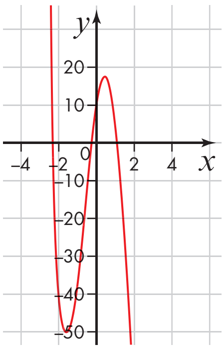
9 a



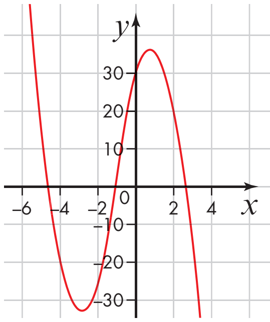
b



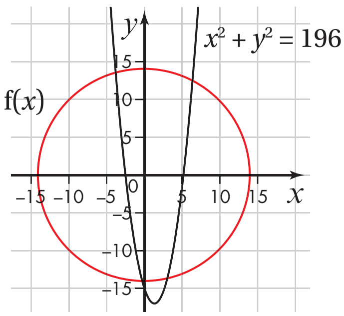
c



d



10 a



b i $f(x) + a$ where $a > 2$.

ii $f(x - a)$ where $a > 9$ or $a < -11$

4 Coordinate geometry 1: Equations of straight lines

Prior Knowledge p 101

1 $(0, 3), (-\frac{3}{2}, 0)$

2 a A: 0, 2

B: 2, 3

C: $-\frac{1}{2}$, 3

D: –2, 3

E: 0, 3

F: –2, $\frac{3}{2}$

b A, E

D, F

c B, C

3 $y = 16 - 8x$

4 $y = 2x - 2$

Exercise 4.1A p 103

1 The lines not in the form $ax + by + c = 0$, where a, b and c are integers are:

c because the right-hand side is not zero

d because the right-hand side is not zero

e because c is not an integer

f because a, b and c are not integers.

2 a Rearrange $y = 4 + 5x$.

$5x - y + 4 = 0$ where $a = 5$, $b = -1$ and $c = 4$

b Rearrange $y = 3 - 2x$.

$2x + y - 3 = 0$ where $a = 2$, $b = 1$ and $c = -3$

3 a Multiply by 3: $3y = x - 21$, and rearrange.

$x - 3y - 21 = 0$ where $a = 1$, $b = -3$ and $c = -21$

b Multiply by 5: $5y = -2x + 30$, and rearrange.

$2x + 5y - 30 = 0$ where $a = 2$, $b = 5$ and $c = -30$

c Multiply by 3: $3y = 4x + \frac{21}{2}$

Multiply by 2: $6y = 8x + 21$, and rearrange.

$8x - 6y + 21 = 0$ where $a = 8$, $b = -6$ and $c = 21$

4 Rearrange: $8x + 3 = 2y$

Rearrange and divide by 2: $y = 4x + \frac{3}{2}$

Gradient = 4 and y intercept = $\frac{3}{2}$; coordinates $(0, \frac{3}{2})$

5 Equation written down incorrectly. It should be $y = 3 - \frac{5}{2}x$.

Not all expressions in the equation have been multiplied by 2. It should be $2y = 6 - 5x$.

Although the values are wrong this step is actually correct. With the correct values it should be $5x + 2y = 6$.

Although the values are wrong this step is actually correct. With the correct values it should be $5x + 2y - 6 = 0$.

6 $\frac{5y}{3} = x - 4$

$5y = 3(x - 4)$

$5y = 3x - 12$

$3x - 5y - 12 = 0$ where $a = 3$, $b = -5$ and $c = -12$

7 When the line intercepts the x axis, $y = 0$.

Substituting $y = 0$:

$-3x + 2 = 0$

$2 = 3x$

$x = \frac{2}{3}$

So the coordinates of the x intercept are $\left(\frac{2}{3}, 0\right)$.

When the line intercepts the y axis, $x = 0$.

Substituting $x = 0$:

$-5y + 2 = 0$

$2 = 5y$

$y = \frac{2}{5}$

So the coordinates of the y intercept are $\left(0, \frac{2}{5}\right)$.

Exercise 4.2A p 105

1 a $y - y_1 = m(x - x_1)$

$y - 0 = 2(x - 3)$

$y = 2x - 6$

b $y - y_1 = m(x - x_1)$

$y - 3 = 3(x - 0)$

$y - 3 = 3x$

$y = 3x + 3$

c $y - y_1 = m(x - x_1)$

$y - 4 = 2(x - 3)$

$y - 4 = 2x - 6$

$y = 2x - 2$

d $y - y_1 = m(x - x_1)$

$y - 3 = -5(x - 2)$

$y - 3 = -5x + 10$

$y = -5x + 13$

2 **a** $y - y_1 = m(x - x_1)$

$y - -5 = -4(x - -2)$

$y + 5 = -4x - 8$

$y = -4x - 13$

b $y - y_1 = m(x - x_1)$

$y - -2 = -1(x - 2)$

$y + 2 = -x + 2$

$y = -x$

3 $y - y_1 = m(x - x_1)$

$y - 3 = -2(x - 0)$

$y - 3 = -2x$

$2x + y - 3 = 0$

4 $y - y_1 = m(x - x_1)$

$y - 0 = 3(x - -1)$

$y = 3x + 3$

$3x - y + 3 = 0$

5 First find the point of intersection.

$2x + 4 = 7 - x$

$3x = 3$

$x = 1$

$y = 2 + 4 = 6$

$m = 3$ and $(x_1, y_1) = (1, 6)$

$y - y_1 = m(x - x_1)$

$y - 6 = 3(x - 1)$

$y - 6 = 3x - 3$

$y = 3x + 3$

6 Find the equation of the first line.

$m = 3$ and $(x_1, y_1) = (1, 1)$

$y - y_1 = m(x - x_1)$

$y - 1 = 3(x - 1)$

$y - 1 = 3x - 3$

$y = 3x - 2$

Find the equation of the second line.

$m = -1$ and $(x_1, y_1) = (4, 6)$

$y - y_1 = m(x - x_1)$

$y - 6 = -1(x - 4)$

$y - 6 = -x + 4$

$y = -x + 10$

Find the point of intersection.

$3x - 2 = -x + 10$

$4x = 12$

$x = 3$

$y = -3 + 10 = 7$

Point of intersection is (3, 7).

7 $m = \frac{5}{2}$ and $(x_1, y_1) = (2, 3)$

$y - y_1 = m(x - x_1)$

$y - 3 = \frac{5}{2}(x - 2)$

When you substitute in $x = 0$, y will equal 0 if the line goes through the origin.

$y - 3 = \frac{5}{2}(0 - 2)$

$y - 3 = \frac{5}{2}(-2)$

$y - 3 = -5$

$y = -2$

The line does not go through the origin.

Exercise 4.3A p 108

1 **a** $m = \frac{y_2 - y_1}{x_2 - x_1}$

$(x_1, y_1) = (2, 3)$

$(x_2, y_2) = (7, 8)$

$m = \frac{8 - 3}{7 - 2}$

$m = \frac{5}{5}$

$m = 1$

b $m = \frac{y_2 - y_1}{x_2 - x_1}$

$(x_1, y_1) = (5, 9)$

$(x_2, y_2) = (3, 3)$

$m = \frac{9 - 3}{5 - 3}$

$m = \frac{6}{2}$

$m = 3$

c $m = \frac{y_2 - y_1}{x_2 - x_1}$

$(x_1, y_1) = (1, -3)$

$(x_2, y_2) = (3, -9)$

$m = \frac{-9 - -3}{3 - 1}$

$m = \frac{-6}{2}$

$m = -3$

2 **a** $m = \frac{y_2 - y_1}{x_2 - x_1}$

$(x_1, y_1) = (8a, 5a)$

$(x_2, y_2) = (3a, 3a)$

$m = \frac{5a - 3a}{8a - 3a}$

$m = \frac{2a}{5a}$

$m = \frac{2}{5}$

b $m = \frac{y_2 - y_1}{x_2 - x_1}$

$(x_1, y_1) = (a, a)$

$(x_2, y_2) = (3a, -5a)$

$m = \frac{-5a - a}{3a - a}$

$m = \frac{-6a}{2a}$

$m = -3$

3 Should be $m = \frac{y_2 - y_1}{x_2 - x_1}$

y coordinate incorrect in (x_2, y_2). It should be (5, –8).

Numerator and denominator confused. Should be:

$m = \frac{-8 - 2}{5 - 1}$

$m = \frac{-10}{4}$

$m = \frac{-5}{2}$

4 **a** $m = \frac{y_2 - y_1}{x_2 - x_1}$

$(x_1, y_1) = (1, 1)$

$(x_2, y_2) = (4, -8)$

$m = \frac{-8 - 1}{4 - 1}$

$m = \frac{-9}{3}$

$m = -3$

Lie on a straight line with a gradient of –3.

b $m = \frac{y_2 - y_1}{x_2 - x_1}$

$(x_1, y_1) = (1, 3)$

$(x_2, y_2) = (4, 9)$

$m = \frac{9 - 3}{4 - 1}$

$m = \frac{6}{3}$

$m = 2$

Do not lie on a straight line with a gradient of –3.

c $m = \frac{y_2 - y_1}{x_2 - x_1}$

$(x_1, y_1) = (1, -7)$

$(x_2, y_2) = (4, -16)$

$m = \frac{-16 - -7}{4 - 1}$

$m = \frac{-9}{3}$

$m = -3$

Lie on a straight line with a gradient of –3.

d $m = \frac{y_2 - y_1}{x_2 - x_1}$

$(x_1, y_1) = (1, 7)$

$(x_2, y_2) = (3, 1)$

$m = \frac{1 - 7}{3 - 1}$

$m = \frac{-6}{2}$

$m = -3$

Lie on a straight line with a gradient of –3.

5 $m = \frac{y_2 - y_1}{x_2 - x_1}$

$(x_1, y_1) = \left(5, \frac{1}{2}\right)$

$(x_2, y_2) = (2, 2)$

$m = \frac{2 - \frac{1}{2}}{2 - 5}$

$m = \frac{\frac{3}{2}}{-3}$

$m = -\frac{1}{2}$

$m = \frac{y_2 - y_1}{x_2 - x_1}$

$(x_1, y_1) = \left(11, -\frac{5}{2}\right)$

$(x_2, y_2) = (2, 2)$

$m = \frac{2 - -\frac{5}{2}}{2 - 11}$

$m = \frac{\frac{9}{2}}{-9}$

$m = -\frac{1}{2}$

The gradient, m, is the same between both of the pairs of points, so we can conclude that all the points lie on a straight line.

6 Ascent:

$m = \frac{y_2 - y_1}{x_2 - x_1}$

$(x_1, y_1) = (-10, 0)$

$(x_2, y_2) = (0, \frac{1}{2})$

$m = \frac{\frac{1}{2} - 0}{0 - -10}$

$m = \frac{\frac{1}{2}}{10}$

$m = \frac{1}{20}$

Descent:

$m = \frac{y_2 - y_1}{x_2 - x_1}$

$(x_1, y_1) = (0, \frac{1}{2})$

$(x_2, y_2) = (2, \frac{3}{8})$

$m = \frac{\frac{3}{8} - \frac{1}{2}}{2 - 0}$

$m = \frac{-\frac{1}{8}}{2}$

$m = -\frac{1}{16}$

The descent is the steepest part because $\frac{1}{16} > \frac{1}{20}$.

7 $m = \frac{y_2 - y_1}{x_2 - x_1}$

$(x_1, y_1) = \left(\frac{1}{2}, \frac{1}{3}\right)$

$(x_2, y_2) = \left(\frac{3}{4}, -\frac{2}{3}\right)$

$m = \frac{-\frac{2}{3} - \frac{1}{3}}{\frac{3}{4} - \frac{1}{2}}$

$m = \frac{-1}{\frac{1}{4}}$

$m = -4$

Exercise 4.4A p 111

1 a $(x_1, y_1) = (2, 3)$

$(x_2, y_2) = (7, 8)$

$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$

$\frac{y - 3}{8 - 3} = \frac{x - 2}{7 - 2}$

$\frac{y - 3}{5} = \frac{x - 2}{5}$

$y - 3 = x - 2$

$y = x + 1$

b $(x_1, y_1) = (3, 3)$

$(x_2, y_2) = (5, 9)$

$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$

$\frac{y - 3}{9 - 3} = \frac{x - 3}{5 - 3}$

$\frac{y - 3}{6} = \frac{x - 3}{2}$

$y - 3 = 3(x - 3)$

$y - 3 = 3x - 9$

$y = 3x - 6$

2 a $(x_1, y_1) = (1, -3)$

$(x_2, y_2) = (3, -9)$

$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$

$\frac{y - -3}{-9 - -3} = \frac{x - 1}{3 - 1}$

$\frac{y + 3}{-6} = \frac{x - 1}{2}$

$y + 3 = -3(x - 1)$

$y + 3 = -3x + 3$

$y = -3x$

b $(x_1, y_1) = (-3, -4)$

$(x_2, y_2) = (-7, 6)$

$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$

$\frac{y - -4}{6 - -4} = \frac{x - -3}{-7 - -3}$

$\frac{y + 4}{10} = \frac{x + 3}{-4}$

$y + 4 = -\frac{5}{2}(x + 3)$

$y + 4 = -\frac{5}{2}x - \frac{15}{2}$

$y = -\frac{5}{2}x - \frac{23}{2}$

3 $\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$

Let $(x_1, y_1) = (1, 2)$

Let $(x_2, y_2) = (5, -8)$

y_1 and y_2 and x_1 and x_2 confused. The next line should be:

$\frac{y - 2}{-8 - 2} = \frac{x - 1}{5 - 1}$

With this correction, the remainder of the calculation should be:

$\frac{y - 2}{-10} = \frac{x - 1}{4}$

$4(y - 2) = -10(x - 1)$

$4y - 8 = -10x + 10$

$4y = -10x + 18$

So $y = \frac{-5x + 9}{2}$

4 $y = 6 - 2x$

y intercept when $x = 0$, so when $y = 6$

$(x_1, y_1) = (0, 6)$

$(x_2, y_2) = (-1, -2)$

$$\frac{y-y_1}{y_2-y_1}=\frac{x-x_1}{x_2-x_1}$$
$$\frac{y-6}{-2-6}=\frac{x-0}{-1-0}$$
$$\frac{y-6}{-8}=-x$$
$$y-6=8x$$
$$8x-y+6=0$$

5 $2x+y-4=0$

x intercept when $y=0$

$2x-4=0$ so $x=2$

$(x_1,y_1)=(2,0)$

$(x_2,y_2)=(3,-7)$

$$\frac{y-y_1}{y_2-y_1}=\frac{x-x_1}{x_2-x_1}$$
$$\frac{y-0}{-7-0}=\frac{x-2}{3-2}$$
$$\frac{y-0}{-7}=\frac{x-2}{1}$$
$$y=-7(x-2)$$
$$y=-7x+14$$
$$7x+y-14=0$$

Gradient $=-7$ and y-intercept (0, 14)

6 Line A:

$(x_1,y_1)=(2,7)$

$(x_2,y_2)=(5,6)$

$$\frac{y-y_1}{y_2-y_1}=\frac{x-x_1}{x_2-x_1}$$
$$\frac{y-7}{6-7}=\frac{x-2}{5-2}$$
$$\frac{y-7}{-1}=\frac{x-2}{3}$$
$$y-7=\frac{2-x}{3}$$
$$y=\tfrac{23}{3}-\tfrac{1}{3}x$$
$$m=-\tfrac{1}{3}$$

Line B:

$(x_1,y_1)=(5,-4)$

$(x_2,y_2)=(7,-6)$

$$\frac{y-y_1}{y_2-y_1}=\frac{x-x_1}{x_2-x_1}$$
$$\frac{y--4}{-6--4}=\frac{x-5}{7-5}$$
$$\frac{y+4}{-2}=\frac{x-5}{2}$$
$$y+4=-x+5$$
$$y=1-x$$
$$m=-1$$

Line B is steeper because $1>\tfrac{1}{3}$. (The negatives can be ignored because they just indicate the direction of the line.)

7 a $(x_1,y_1)=\left(\tfrac{1}{2},\tfrac{1}{3}\right)$

$(x_2,y_2)=\left(\tfrac{3}{4},-\tfrac{2}{3}\right)$

$$\frac{y-y_1}{y_2-y_1}=\frac{x-x_1}{x_2-x_1}$$
$$\frac{y-\frac{1}{3}}{-\frac{2}{3}-\frac{1}{3}}=\frac{x-\frac{1}{2}}{\frac{3}{4}-\frac{1}{2}}$$
$$\frac{y-\frac{1}{3}}{-1}=\frac{x-\frac{1}{2}}{\frac{1}{4}}$$
$$y-\tfrac{1}{3}=-4\left(x-\tfrac{1}{2}\right)$$
$$y-\tfrac{1}{3}=-4x+2$$
$$y=-4x+\tfrac{7}{3}$$

b $(x_1,y_1)=\left(\tfrac{2}{7},-\tfrac{1}{5}\right)$

$(x_2,y_2)=\left(-\tfrac{1}{3},-\tfrac{1}{2}\right)$

$$\frac{y-y_1}{y_2-y_1}=\frac{x-x_1}{x_2-x_1}$$
$$\frac{y--\frac{1}{5}}{-\frac{1}{2}--\frac{1}{5}}=\frac{x-\frac{2}{7}}{-\frac{1}{3}-\frac{2}{7}}$$
$$\frac{y+\frac{1}{5}}{-\frac{5}{10}+\frac{2}{10}}=\frac{x-\frac{2}{7}}{-\frac{7}{21}-\frac{6}{21}}$$
$$\frac{y+\frac{1}{5}}{-\frac{3}{10}}=\frac{x-\frac{2}{7}}{-\frac{13}{21}}$$
$$\frac{y+\frac{1}{5}}{-\frac{3}{10}}=-\tfrac{21}{13}\left(x-\tfrac{2}{7}\right)$$
$$y+\tfrac{1}{5}=\tfrac{63}{130}\left(x-\tfrac{2}{7}\right)$$
$$y+\tfrac{1}{5}=\tfrac{63}{130}x-\tfrac{9}{65}$$
$$y=\tfrac{63}{130}x-\tfrac{22}{65}$$

8 $2x-5y+7=0$

y intercept when $x=0$

$-5y+7=0$ so $y=\tfrac{7}{5}$

$y=-3x+5$

x intercept when $y=0$

$-3x+5=0$ so $x=\tfrac{5}{3}$

$(x_1,y_1)=\left(0,\tfrac{7}{5}\right)$

$(x_2,y_2)=\left(\tfrac{5}{3},0\right)$

$$\frac{y-y_1}{y_2-y_1}=\frac{x-x_1}{x_2-x_1}$$
$$\frac{y-\frac{7}{5}}{0-\frac{7}{5}}=\frac{x-0}{\frac{5}{3}-0}$$
$$\frac{y-\frac{7}{5}}{-\frac{7}{5}}=\tfrac{3}{5}x$$

$y-\frac{7}{5}=-\frac{7}{5}\left(\frac{3}{5}x\right)$

$y-\frac{7}{5}=-\frac{21}{25}x$

$y=-\frac{21}{25}x+\frac{7}{5}$

Rearranging: $21x+25y-35=0$

9 Line A:

$(x_1,y_1)=(3,5)$

$(x_2,y_2)=(4,9)$

$\frac{y-y_1}{y_2-y_1}=\frac{x-x_1}{x_2-x_1}$

$\frac{y-5}{9-5}=\frac{x-3}{4-3}$

$\frac{y-5}{4}=\frac{x-3}{1}$

$y-5=4(x-3)$

$y-5=4x-12$

$y=4x-7$

Line B:

$(x_1,y_1)=(1,-3)$

$(x_2,y_2)=(5,-31)$

$\frac{y-y_1}{y_2-y_1}=\frac{x-x_1}{x_2-x_1}$

$\frac{y--3}{-31--3}=\frac{x-1}{5-1}$

$\frac{y+3}{-28}=\frac{x-1}{4}$

$y+3=-7(x-1)$

$y+3=-7x+7$

$y=4-7x$

At the point of intersection:

$4x-7=4-7x$

$11x=11$

$x=1$

Substituting into either equation gives $y=-3$. So point of intersection is $(1,-3)$.

10 Point A:

$x-9=5-x$

$2x=14$

$x=7, y=-2$

Point B:

$7-3x=2x+8$

$5x=-1$

$x=-\frac{1}{5}, y=\frac{38}{5}$

$(x_1,y_1)=(7,-2)$

$(x_2,y_2)=\left(-\frac{1}{5},\frac{38}{5}\right)$

$\frac{y-y_1}{y_2-y_1}=\frac{x-x_1}{x_2-x_1}$

$\frac{y--2}{\frac{38}{5}--2}=\frac{x-7}{-\frac{1}{5}-7}$

$\frac{y+2}{\frac{48}{5}}=\frac{x-7}{-\frac{36}{5}}$

$-\frac{36}{5}(y+2)=\frac{48}{5}(x-7)$

$y+2=-\frac{4}{3}(x-7)$

$y+2=-\frac{4}{3}x+\frac{28}{3}$

$y=-\frac{4}{3}x+\frac{22}{3}$

When $x=1$, $y=6$ so this line does not pass through the point $(1, 2)$.

Exercise 4.5A

p 115

1 **a** $m=2$ in both equations so the lines are parallel.

b $m=-3$ in both equations so the lines are parallel.

2 Line A:

$(x_1,y_1)=(1,7)$

$(x_2,y_2)=(3,11)$

$m=\frac{y_2-y_1}{x_2-x_1}$

$m=\frac{11-7}{3-1}$

$m=2$

Line B:

$(x_1,y_1)=(2,-3)$

$(x_2,y_2)=(5,3)$

$m=\frac{y_2-y_1}{x_2-x_1}$

$m=\frac{3--3}{5-2}$

$m=2$

$m=2$ for both, so the lines are parallel.

3 From $y=7x-2$, $m=7$.

The x intercept of $y=x-3$ is when $y=0$ so is at $(3, 0)$.

$(x_1,y_1)=(3,0)$

$y-y_1=m(x-x_1)$

$y-0=7(x-3)$

$y=7x-21$

4 a Arrange the equations in the form $y = mx + c$:
$y = 4x + 2$, $y = -4x - 3$
m is not equal so the lines are not parallel.

b Arrange the equations in the form $y = mx + c$:
$y = 3x + 1$, $y = 3x - 3$
$m = 3$ in both equations so the lines are parallel.

5 Line A:
$(x_1, y_1) = (-1, 3)$
$(x_2, y_2) = (3, -1)$
$$m = \frac{y_2 - y_1}{x_2 - x_1}$$
$$m = \frac{-1 - 3}{3 - -1}$$
$$m = -1$$
Line B:
$(x_1, y_1) = (-2, 2)$
$(x_2, y_2) = (-3, 3)$
$$m = \frac{y_2 - y_1}{x_2 - x_1}$$
$$m = \frac{3 - 2}{-3 - -2}$$
$$m = -1$$
$m = -1$ for both, so the lines are parallel.

6 From $-4x + y + 7 = 0$, $y = 4x - 7$, so $m = 4$.
The y intercept of $2x - y + 3 = 0$ is when $x = 0$ so is at (0, 3).
$(x_1, y_1) = (0, 3)$
$$y - y_1 = m(x - x_1)$$
$$y - 3 = 4(x - 0)$$
$$y = 4x + 3$$

7 a Arrange the equations in the form $y = mx + c$:
$y = \frac{5}{3}x - \frac{7}{3}$, $y = \frac{5}{3}x + \frac{1}{2}$
$m = \frac{5}{3}$ in both equations so the lines are parallel.

b Arrange the equations in the form $y = mx + c$:
$y = \frac{8}{3}x - 7$, $y = \frac{4}{3}x - \frac{1}{3}$
m is not equal so the lines are not parallel.

8 Arrange the equations in the form $y = mx + c$:
$2x - y + 6 = 0$, $y = 2x + 6$, $m = 2$
$2x + 4y - 44 = 0$, $4y = 44 - 2x$, $y = 11 - \frac{1}{2}x$, $m = -\frac{1}{2}$
$2x - y - 4 = 0$, $y = 2x - 4$, $m = 2$
$2x + 4y - 24 = 0$, $4y = 24 - 2x$, $y = 6 - \frac{1}{2}x$, $m = -\frac{1}{2}$
So $2x - y + 6 = 0$ and $2x - y - 4 = 0$ are parallel lines as $m = 2$ for both lines.
And $2x + 4y - 44 = 0$ and $2x + 4y - 24 = 0$ are parallel lines as $m = -\frac{1}{2}$ for both lines.

Exercise 4.5B p 117

1 a $m_1 = 2$ and $m_2 = \frac{1}{2}$ so $m_1m_2 \neq -1$ so the lines are not perpendicular.

b $m_1 = -3$ and $m_2 = \frac{1}{3}$ so $m_1m_2 = -1$ so the lines are perpendicular.

2 $m_1 = -\frac{1}{2}$ for the given line but the gradient of a line perpendicular to this will be $m_2 = 2$.
Let $(x_1, y_1) = (1, 0)$.
$$y - y_1 = m(x - x_1)$$
(x_1, y_1) incorrectly substituted. Correcting this and using correct value of m_2 gives:
$y - 0 = 2x - 2$
So $y = 2x - 2$

3 a $m_1 = 3$ and $m_2 = -\frac{1}{3}$ so $m_1m_2 = -1$ so the lines are perpendicular.

b $m_1 = 3$ and $m_2 = 12$ so $m_1m_2 \neq -1$ so the lines are not perpendicular.

4 Line A:
$(x_1, y_1) = (1, 1)$
$(x_2, y_2) = (3, 9)$
$$m = \frac{y_2 - y_1}{x_2 - x_1}$$
$$m = \frac{9 - 1}{3 - 1}$$
$$m = 4$$
Line B:
$(x_1, y_1) = (4, 0)$
$(x_2, y_2) = (8, -1)$
$$m = \frac{y_2 - y_1}{x_2 - x_1}$$
$$m = \frac{-1 - 0}{8 - 4}$$
$$m = -\frac{1}{4}$$
$m_1 = 4$ and $m_2 = -\frac{1}{4}$ so $m_1m_2 = -1$ so the lines are perpendicular.

5 Arrange the equations in the form $y = mx + c$:
$2x - y + 6 = 0$, $y = 2x + 6$, $m = 2$
$2x + 4y - 44 = 0$, $4y = 44 - 2x$, $y = 11 - \frac{1}{2}x$, $m = -\frac{1}{2}$
$2x - y - 4 = 0$, $y = 2x - 4$, $m = 2$
$2x + 4y - 24 = 0$, $4y = 24 - 2x$, $y = 6 - \frac{1}{2}x$, $m = -\frac{1}{2}$

So $2x - y + 6 = 0$ and $2x + 4y - 44 = 0$ are perpendicular lines as $m_1m_2 = -1$.
And $2x - y + 6 = 0$ and $2x + 4y - 24 = 0$ are perpendicular lines as $m_1m_2 = -1$.
And $2x - y - 4 = 0$ and $2x + 4y - 44 = 0$ are perpendicular lines as $m_1m_2 = -1$.
And $2x - y - 4 = 0$ and $2x + 4y - 24 = 0$ are perpendicular lines as $m_1m_2 = -1$.

6 From $8x - 2y - 7 = 0$, $8x - 7 = 2y$, $4x - \frac{7}{2} = y$, $m_1 = 4$

$m_1m_2 = -1$

So $m_2 = -\frac{1}{4}$

The y intercept of $4x - 2y + 5 = 0$ is when $x = 0$ so is at $(0, \frac{5}{2})$.

$$(x_1, y_1) = (0, \tfrac{5}{2})$$
$$y - y_1 = m(x - x_1)$$
$$y - \tfrac{5}{2} = -\tfrac{1}{4}(x - 0)$$
$$y = \tfrac{5}{2} - \tfrac{1}{4}x$$

7 **a** $m_1 = \frac{5}{3}$ and $m_2 = -\frac{3}{5}$ so $m_1m_2 = -1$ so the lines are perpendicular.

b $m_1 = \frac{9}{10}$ and $m_2 = \frac{10}{9}$ so $m_1m_2 \neq -1$ so the lines are not perpendicular.

8 Describe as line A:

$(x_1, y_1) = (0, 6)$

$(x_2, y_2) = (4, 4)$

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$
$$m = \frac{4 - 6}{4 - 0}$$
$$m = -\frac{2}{4}$$
$$m = -\frac{1}{2}$$

Describe as line B:

$(x_1, y_1) = (0, 6)$

$(x_2, y_2) = (2, 10)$

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$
$$m = \frac{10 - 6}{2 - 0}$$
$$m = \frac{4}{2}$$
$$m = 2$$

Describe as line C:

$(x_1, y_1) = (6, 8)$

$(x_2, y_2) = (2, 10)$

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$
$$m = \frac{10 - 8}{2 - 6}$$
$$m = \frac{2}{-4}$$
$$m = -\frac{1}{2}$$

Describe as line D:

$(x_1, y_1) = (4, 4)$

$(x_2, y_2) = (6, 8)$

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$
$$m = \frac{8 - 4}{6 - 4}$$
$$m = \frac{4}{2}$$
$$m = 2$$

So A is parallel to C, B is parallel to D, A is perpendicular to B and D, and C is perpendicular to B and D.

So the quadrilateral can only be a square or a rectangle.

Exercise 4.6A p 122

1 $(x_1, y_1) = (100, 160)$

$(x_2, y_2) = (500, 800)$

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$$
$$\frac{y - 160}{800 - 160} = \frac{x - 100}{500 - 100}$$
$$\frac{y - 160}{640} = \frac{x - 100}{400}$$
$$y - 160 = \tfrac{640}{400}(x - 100)$$
$$y - 160 = \tfrac{8}{5}x - 160$$
$$y = \tfrac{8}{5}x$$

2 **a** False: $m_1 = 2$ and $m_2 = -2$ so $m_1m_2 \neq -1$ so the lines are not perpendicular.

b False: $c_1 = 6$ and $c_2 = 12$ so the lines do not share the same y intercept.

c False: negative gradient means the object is travelling back to the start.

d True: speed is the gradient and the gradients are the same (the difference in the signs just indicates direction) so they are travelling at the same speed.

e False: $m_1 \neq m_2$ so the lines are not parallel.

f False: positive gradient means the object is travelling away from the start.

3 We have been given two pairs of coordinates for each printer. The cost of the printer is actually the start of the costs when $x = 0$, i.e. the y intercept. We have also been given the y coordinates when $x = 5$.

Anna's printer:

$(x_1, y_1) = (0, 50)$

$(x_2, y_2) = (5, 950)$

$$\frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1}$$

$$\frac{y-50}{950-50} = \frac{x-0}{5-0}$$

$$\frac{y-50}{900} = \frac{x}{5}$$

$$y - 50 = 180x$$

$$y = 180x + 50$$

Bhavini's printer:

$(x_1, y_1) = (0, 67.50)$

$(x_2, y_2) = (5, 667.50)$

$$\frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1}$$

$$\frac{y-67.50}{667.50-67.50} = \frac{x-0}{5-0}$$

$$\frac{y-67.50}{600} = \frac{x}{5}$$

$$y - 67.50 = 120x$$

$$y = 120x + 67.50$$

m is the cost per year to run the printer and is £180 in Anna's case and £120 in Bhavini's case.

c is the cost to buy the printer and is £50 in Anna's case and £67.50 in Bhavini's case.

The cost of one ink cartridge for Anna's printer is £180 ÷ 4 = £45.

The cost of one ink cartridge for Bhavini's printer is £120 ÷ 4 = £30.

Nothing is significant about the x intercepts. Although you can mathematically work out the x intercept in each case, this would actually be a negative value which would not make sense as you cannot have a negative number of years.

4 Taxi firm A:

$(x_1, y_1) = (0, 0)$

$(x_2, y_2) = (10, 50)$

$$\frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1}$$

$$\frac{y-0}{50-0} = \frac{x-0}{10-0}$$

$$10y = 50x$$

$$y = 5x$$

When $x = 4$, $y = 20$.

When $x = 7$, $y = 35$.

Taxi firm B:

$(x_1, y_1) = (0, 10)$

$(x_2, y_2) = (10, 40)$

$$\frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1}$$

$$\frac{y-10}{40-10} = \frac{x-0}{10-0}$$

$$10y - 100 = 30x$$

$$y = 3x + 10$$

When $x = 4$, $y = 22$.

When $x = 7$, $y = 31$.

Taxi firm C:

$(x_1, y_1) = (0, 15)$

$(x_2, y_2) = (10, 35)$

$$\frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1}$$

$$\frac{y-15}{35-15} = \frac{x-0}{10-0}$$

$$\frac{y-15}{20} = \frac{x}{10}$$

$$10y - 150 = 20x$$

$$y = 2x + 15$$

When $x = 4$, $y = 23$.

When $x = 7$, $y = 29$.

Taxi firm A is the cheapest on a 4-mile journey.

Taxi firm C is the cheapest on a 7-mile journey.

5 $(x_1, y_1) = (0.57, 1)$

$(x_2, y_2) = (1, 1.75)$

$$\frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1}$$

$$\frac{y-1}{1.75-1} = \frac{x-0.57}{1-0.57}$$

$$\frac{y-1}{0.75} = \frac{x-0.57}{0.43}$$

$$y - 1 = \tfrac{75}{43}(x - 0.57)$$

$$y - 1 = \tfrac{75}{43}x - \tfrac{4275}{4300}$$

$$y = \tfrac{75}{43}x + \tfrac{1}{172}$$

Exam-style questions 4 p 124

1 $m = 3$, $(x_1, y_1) = (1, 6)$

$$y - y_1 = m(x - x_1)$$
$$y - 6 = 3(x - 1)$$
$$y - 6 = 3x - 3$$
$$y = 3x + 3$$

2 **a** Line A:

$(x_1, y_1) = (0, 7)$

$(x_2, y_2) = (2, 3)$

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$
$$m = \frac{3 - 7}{2 - 0}$$
$$m = -\frac{4}{2}$$
$$m = -2$$

Line B:

$(x_1, y_1) = (6, 4)$

$(x_2, y_2) = (8, 5)$

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$
$$m = \frac{5 - 4}{8 - 6}$$
$$m = \frac{1}{2}$$

b Line A is steeper because $2 > \frac{1}{2}$.

3 $m_1 = 1$

$m_1 m_2 = -1$

so $m_2 = -1$

$(x_1, y_1) = (0, 5)$

$$y - y_1 = m(x - x_1)$$
$$y - 5 = -1(x - 0)$$
$$y = -x + 5$$

4 **a** Runner A starts 3 miles away from runner B's home.

The two runners meet 9 miles away from runner B's home so runner A will have run 6 miles.

b Runner A:

$(x_1, y_1) = (0, 3)$

$(x_2, y_2) = (3, 9)$

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$$
$$\frac{y - 3}{9 - 3} = \frac{x - 0}{3 - 0}$$
$$\frac{y - 3}{6} = \frac{x}{3}$$
$$y - 3 = 2x$$
$$y = 2x + 3$$

Runner B:

$(x_1, y_1) = (0, 0)$

$(x_2, y_2) = (3, 9)$

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$$
$$\frac{y - 0}{9 - 0} = \frac{x - 0}{3 - 0}$$
$$\frac{y}{9} = \frac{x}{3}$$
$$y = 3x$$

c Runner B runs faster. Gradient = speed and $3 > 2$.

5 **a** $m_1 = 2$

$m_1 m_2 = -1$

so $m_2 = -\frac{1}{2}$

$(x_1, y_1) = (12, 4)$

$$y - y_1 = m(x - x_1)$$
$$y - 4 = -\frac{1}{2}(x - 12)$$
$$y - 4 = -\frac{1}{2}x + 6$$
$$y = -\frac{1}{2}x + 10$$

b When $x = 0$, $y = 10$

y intercept $= (0, 10)$

When $y = 0$, $x = 20$

x intercept $= (20, 0)$

c Area of triangle $= \frac{1}{2}bh = \frac{1}{2} \times 20 \times 10$

$= 100 \text{ units}^2$

6 Line AB:

$(x_1, y_1) = (-2, 8)$

$(x_2, y_2) = (4, 5)$

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$
$$m = \frac{5 - 8}{4 - -2}$$
$$m_1 = -\frac{1}{2}$$

Line BC:

$(x_1, y_1) = (-2, 8)$

$(x_2, y_2) = (-10, -8)$

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m = \frac{-8 - 8}{-10 - -2}$$

$$m_2 = 2$$

$$m_1 m_2 = \left(-\tfrac{1}{2}\right)(2) = -1$$

So AB and BC are perpendicular to each other and ABC is a right-angled triangle.

7 Pay as you go:

$(x_1, y_1) = (0, 5)$

$(x_2, y_2) = (1, 7)$

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$$

$$\frac{y - 5}{7 - 5} = \frac{x - 0}{1 - 0}$$

$$\frac{y - 5}{2} = x$$

$$y = 2x + 5$$

For this option:

4 rides: $x = 4$ so $y = £13$

17 rides: $x = 17$ so $y = £39$

For other option (book of tickets):

Cost of 4 rides = £5 + £10 = £15

so pay as you go option is cheaper.

Cost of 17 rides = £5 + (3 × £10) = £35

so book of tickets option is cheaper.

8 Line A:

$(x_1, y_1) = (8, 5)$

$(x_2, y_2) = (-12, a)$

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$\frac{1}{2} = \frac{a - 5}{-12 - 8}$$

$$-20 = 2a - 10$$

$$a = -5$$

$m = -2$, $(x_1, y_1) = (0, -5)$

$$y - \ -5 = -2(x - 0)$$

$$y = -2x - 5$$

9 $(x_1, y_1) = (2\sqrt{2}, \sqrt{2})$

$(x_2, y_2) = (\sqrt{2}, 2\sqrt{3})$

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m = \frac{2\sqrt{3} - \sqrt{2}}{\sqrt{2} - 2\sqrt{2}}$$

$$m = \frac{2\sqrt{3} - \sqrt{2}}{-\sqrt{2}}$$

Gradient perpendicular to this:

$$m = \frac{\sqrt{2}}{2\sqrt{3} - \sqrt{2}}$$

$(x_1, y_1) = (0, 3)$

$$y - y_1 = m(x - x_1)$$

$$y - 3 = \frac{\sqrt{2}}{2\sqrt{3} - \sqrt{2}}(x - 0)$$

$$y = \frac{2\sqrt{6} + 2}{10}x + 3$$

10 $x + 4y - 8 = 0$

$x = 8 - 4y$

$3(8 - 4y) + 5y + 15 = 0$

Solving gives $y = \frac{39}{7}$

Substituting this value back into the original equation gives $x = -\frac{100}{7}$

Point of intersection $(x_1, y_1) = \left(-\frac{100}{7}, \frac{39}{7}\right)$

$3x + 5y + 15 = 0$

Rearranging gives $m_1 = -\frac{3}{5}$

So $m_2 = \frac{5}{3}$

$$y - y_1 = m(x - x_1)$$

$$y - \frac{39}{7} = \frac{5}{3}\left(x - -\frac{100}{7}\right)$$

$$y = \frac{5}{3}x + \frac{617}{21}$$

11 a Line A:

$(x_1, y_1) = (0, 32)$

$(x_2, y_2) = (5, 41)$

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$$

$$\frac{y - 32}{41 - 32} = \frac{x - 0}{5 - 0}$$

$$\frac{y - 32}{9} = \frac{x}{5}$$

$$y = \frac{9}{5}x + 32$$

Line B:

$(x_1, y_1) = (32, 0)$

$(x_2, y_2) = (41, 5)$

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$$

$$\frac{y - 0}{5 - 0} = \frac{x - 32}{41 - 32}$$

$$\frac{y}{5} = \frac{x - 32}{9}$$

$$y = \tfrac{5}{9}(x - 32)$$

The two equations are the inverses of each other.

b At the point of intersection:

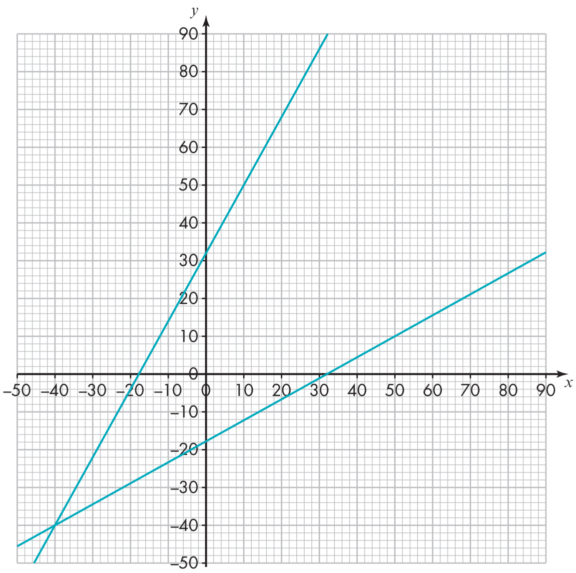
$\tfrac{9}{5}x + 32 = \tfrac{5}{9}(x - 32)$

Rearranging gives $x = -40$

Substituting $x = -40$ back into one of the original equations gives $y = -40$.

So the point of intersection is $(-40, -40)$.

c The transformation linking the two lines is a reflection in the line $y = x$.



d The point of intersection is the only point where the temperature in degrees Celsius and the temperature in degrees Fahrenheit have the same numerical value, i.e. at $(-40, -40)$.

5 Coordinate geometry 2: Circles

Prior knowledge p 127

1 Gradient $= \dfrac{9 - 5}{(-2) - 8} = \dfrac{4}{-10} = -\dfrac{2}{5}$

2 Gradient of first line $= -3$

Perpendicular gradient $= \dfrac{1}{3}$

Equation of perpendicular line through (5, 8) is given by $y - 8 = \dfrac{1}{3}(x - 5)$.

$3y - 24 = x - 5$

$x - 3y + 19 = 0$

3 Substitute $y = \dfrac{2}{3}x - 2$ into $x + 2y = 17$.

$x + 2(\dfrac{2}{3}x - 2) = 17$

$x + \dfrac{4}{3}x - 4 = 17$

$\dfrac{7}{3}x = 21$

$7x = 63$

$x = 9$

When $x = 9$, $y = \dfrac{2}{3} \times 9 - 2 = 4$

Hence point of intersection $= (9, 4)$.

4 Complete square on $x^2 - 14x + 23$.

$x^2 - 14x + 23$

$= (x - 7)^2 - 49 + 23$

$= (x - 7)^2 - 26$

Exercise 5.1A p 131

1 **a** Centre $= (-5, 8)$, radius $= 6$

b Centre $= (19, 33)$, radius $= 20$

c Centre $= (0, -4)$, radius $= 3\sqrt{5}$

d Centre $= (-3, -10)$, radius $= 2\sqrt{7}$

2 **a** $(x + 5)^2 + (y - 9)^2 = 49$ and $x^2 + y^2 + 10x - 18y + 57 = 0$

b $(x + 11)^2 + (y + 1)^2 = 169$ and $x^2 + y^2 + 22x + 2y - 47 = 0$

c $(x - 3)^2 + y^2 = 48$ and $x^2 + y^2 - 6x - 39 = 0$

d $(x - 14)^2 + (y - 6)^2 = 44$ and $x^2 + y^2 - 28x - 12y + 188 = 0$

3 **a** $(x - 5)^2 + (y + 3)^2 = 16$

b $(x + 4)^2 + (y - 2)^2 = 9$

c $(x + 1)^2 + (y - 4)^2 = 20$

4 **a** $(x - 9)^2 - 81 + (y + 7)^2 - 49 = 14$

$(x - 9)^2 + (y + 7)^2 = 144$

Centre $= (9, -7)$, radius $= 12$

b $(x + 4)^2 - 16 + y^2 = 9$

$(x + 4)^2 + y^2 = 25$

Centre $= (-4, 0)$, radius $= 5$

c $x^2 + y^2 + 10x + 18y + 79 = 0$

$(x + 5)^2 - 25 + (y + 9)^2 - 81 + 79 = 0$

$(x + 5)^2 + (y + 9)^2 = 27$

Centre = (–5, –9), radius = $3\sqrt{3}$

d $x^2 + y^2 + 30x - 6y + \frac{855}{4} = 0$

$(x + 15)^2 - 225 + (y - 3)^2 - 9 + \frac{855}{4} = 0$

$(x + 15)^2 + (y - 3)^2 = \frac{81}{4}$

Centre = (–15, 3), radius = $\frac{9}{2}$

5 $d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$

$r^2 = (-4 - 1)^2 + (7 - 3)^2$

$r^2 = (-5)^2 + (4)^2 = 41$

The circle has the equation $(x + 4)^2 + (y - 7)^2 = 41$.

6 $(x - 9)^2 + (y - 2)^2 = 81$

7 $d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$

$d^2 = (4 - 1)^2 + (7 - 3)^2$

$d^2 = (3)^2 + (4)^2 = 25$

The circle has the equation $(x - 1)^2 + (y - 3)^2 = 25$.

The locations on the x-axis will lie on the circumference of the circle.

When $y = 0$, $(x - 1)^2 + (-3)^2 = 25$

$(x - 1)^2 + 9 = 25$

$(x - 1)^2 = 16$

$x - 1 = \pm 4$

$x = 1 \pm 4 = 5$ or -3

The circle cuts the x-axis at (5, 0) and (–3, 0).

8 Rewrite the equation of the circle by completing the square.

$(x + \frac{1}{2}p)^2 - \frac{1}{4}p^2 + (y + 3)^2 - 9 = 96$

$(x + \frac{1}{2}p)^2 + (y + 3)^2 = \frac{1}{4}p^2 + 9 + 96$

Since $r^2 = 121$, $\frac{1}{4}p^2 + 9 + 96 = 121$

$\frac{1}{4}p^2 + 9 + 96 = 121$

$\frac{1}{4}p^2 = 16$

$p^2 = 64$

Since p is a positive constant, $p = \sqrt{64} = 8$.

The centre of the circle has the coordinates $(-\frac{1}{2}p, -3)$.

Since $p = 8$, the centre has the coordinates (–4, –3).

The centre is 5 units from the origin.

Exercise 5.1B p 133

1 $d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$

$d^2 = (-17 - 27)^2 + (25 - (-8))^2$

$d^2 = (-44)^2 + (33)^2 = 3025$

$d = \sqrt{3025} = 55$

2 a The centre is the midpoint between D and E.

Centre = $\left(\frac{2+14}{2}, \frac{9-7}{2}\right) = (8, 1)$

b The radius is the distance between the centre and a point on the circumference.

Let the centre $(8, 1) = (x_1, y_1)$ and $D(2, 9) = (x_2, y_2)$.

$d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$

$r^2 = (8 - 2)^2 + (1 - 9)^2$

$r^2 = (6)^2 + (-8)^2 = 100$

$r = \sqrt{100} = 10$

c The circle has the equation $(x - 8)^2 + (y - 1)^2 = 100$.

3 Centre of circle = $\left(\frac{-3+21}{2}, \frac{8-2}{2}\right) = (9, 3)$

$r^2 = (9 - (-3))^2 + (3 - 8)^2$

$r^2 = (12)^2 + (-5)^2 = 169$

The circle has the equation $(x - 9)^2 + (y - 3)^2 = 169$.

4 a The student has correctly worked out that $AB^2 = 1369$, but the equation of the circle needs the square of the radius, not the square of the diameter. The diameter is $\sqrt{1369} = 37$, so the radius is $\frac{37}{2}$.

b $(x + 0.5)^2 + (y - 17)^2 = (\frac{37}{2})^2$

5 a $\left(\frac{x+9}{2}, \frac{y+5}{2}\right) = (5, -3)$

$x + 9 = 2 \times 5 = 10$

$x = 1$

$y + 5 = 2 \times -3 = -6$

$y = -11$

b $d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$

$r^2 = (9 - 5)^2 + (5 - (-3))^2$

$r^2 = (4)^2 + (8)^2 = 80$

$r = \sqrt{80} = 4\sqrt{5}$

c $(x - 5)^2 + (y + 3)^2 = 80$

d Substitute the coordinates of H into the equation of the circle.

$(-3 - 5)^2 + (1 + 3)^2 = (-8)^2 + (4)^2 = 80$

Since both sides of the equation are satisfied, H lies on C.

6 a The centre of the circle is the midpoint of ST = $\left(\frac{-9+5}{2}, \frac{4+10}{2}\right) = (-2, 7)$.

The midpoint of UV will also be the centre of the circle, (–2, 7).

Hence $\left(\frac{1+p}{2}, \frac{14+q}{2}\right) = (-2, 7)$

$1 + p = 2 \times -2 = -4$

$p = -5$

$14 + q = 2 \times 7 = 14$

$q = 0$

b $r^2 = (-9 - (-2))^2 + (4 - 7)^2$

$r^2 = (-7)^2 + (-3)^2 = 58$

The circle has the equation $(x + 2)^2 + (y - 7)^2 = 58$.

7 The equation of the circle can be rewritten as $(x + 3)^2 + (y - 8)^2 = 125$, so $(-3, 8)$ is the centre of the circle.

Substitute $y = 2x + 14$ into $x^2 + y^2 + 6x - 16y = 52$.

$x^2 + (2x + 14)^2 + 6x - 16(2x + 14) = 52$

Expand and simplify.

$x^2 + 4x^2 + 56x + 196 + 6x - 32x - 224 = 52$

$5x^2 + 30x - 80 = 0$

$x^2 + 6x - 16 = 0$

Solve.

$(x + 8)(x - 2) = 0$

$x = -8$ or 2

M and N are the two intersection points, it doesn't matter which is which.

When $x = -8$, $y = 2 \times -8 + 14 = -2$, so M = $(-8, -2)$

When $x = 2$, $y = 2 \times 2 + 14 = 18$, so N = $(2, 18)$

The midpoint of MN = $\left(\frac{-8+2}{2}, \frac{-2+18}{2}\right) = (-3, 8)$

Since the midpoint of MN is the centre of the circle, MN is a diameter of the circle.

Exercise 5.2A p 137

1 a $d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$

$DE^2 = (-1 - 1)^2 + (11 - 5)^2$

$DE^2 = (-2)^2 + (6)^2 = 40$

$EF^2 = (1 - 13)^2 + (5 - 9)^2$

$EF^2 = (-12)^2 + (-4)^2 = 160$

$DF^2 = (-1 - 13)^2 + (11 - 9)^2$

$DF^2 = (-14)^2 + (2)^2 = 200$

Since $40 + 160 = 200$, $DE^2 + EF^2 = DF^2$.

b Gradient of DE = $\frac{11-5}{-1-1} = -3$

Gradient of EF = $\frac{9-5}{13-1} = \frac{1}{3}$

Since $m_1 \times m_2 = -1$, DE and EF are perpendicular.

c DF is a diameter of the circle.

Centre = $\left(\frac{-1+13}{2}, \frac{11+9}{2}\right) = (6, 10)$

For the radius:

$d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$

$r^2 = (6 - 1)^2 + (10 - 5)^2$

$r^2 = (5)^2 + (5)^2 = 50$

The equation of the circle is $(x - 6)^2 + (y - 10)^2 = 50$.

2 a Prove using Pythagoras' theorem (could compare gradients).

$PQ^2 = (-5 - 11)^2 + (-4 - 8)^2 = (-16)^2 + (-12)^2 = 400$

$PR^2 = (-5 - 13)^2 + (-4 - 2)^2 = (-18)^2 + (-6)^2 = 360$

$QR^2 = (11 - 13)^2 + (8 - 2)^2 = (-2)^2 + (6)^2 = 40$

Since $PR^2 + QR^2 = PQ^2$, Pythagoras' theorem is satisfied and the triangle PQR is right-angled.

b Centre = $\left(\frac{-5+11}{2}, \frac{-4+8}{2}\right) = (3, 2)$

For the radius:

$d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$

$r^2 = (3 - 11)^2 + (2 - 8)^2$

$r^2 = (-8)^2 + (-6)^2 = 100$

Area of triangle = $\frac{1}{2}bh = \frac{1}{2} \times \sqrt{360} \times \sqrt{40} = 60$

Area of circle = $\pi r^2 = \pi \times 10^2 = 100\pi$

Green area = $(100\pi - 60)$ m^2

c $100\pi - 60 = 254.16$ m^2

$254.16 \div 5 = 50.8$, so 51 tins are required.

$51 \times 4 = £204$

3 Since AB is a diameter, ACB = 90° and the lengths of the sides satisfy Pythagoras' theorem.

$AC^2 + BC^2 = AB^2$

$(4x - 9)^2 + (14 - x)^2 = (2x + 5)^2$

$16x^2 - 72x + 81 + 196 - 28x + x^2 = 4x^2 + 20x + 25$

$13x^2 - 120x + 252 = 0$

$(13x - 42)(x - 6) = 0$

$x = \frac{42}{13}$ or 6

When $x = \frac{42}{13}$, $d = 2 \times \frac{42}{13} + 5 = \frac{149}{13}$

Radius = $\frac{1}{2} \times \frac{149}{13} = \frac{149}{26}$

When $x = 6$, $d = 2 \times 6 + 5 = 17$

Radius = $\frac{1}{2} \times 17 = \frac{17}{2}$

4 a Substitute $x = 1$ and $y = 0$ into $(x + 2)^2 + (y - 1)^2 = r^2$.

$(1 + 2)^2 + (0 - 1)^2 = r^2$

$(3)^2 + (-1)^2 = r^2$

$r^2 = 10$

b Substitute $x = -3$ and $y = q$ into $(x + 2)^2 + (y - 1)^2 = 10$

$(-3 + 2)^2 + (q - 1)^2 = 10$

$(-1)^2 + (q - 1)^2 = 10$

$(q - 1)^2 = 9$

$q - 1 = \pm 3$

$q = 1 \pm 3$

$q = -2$ or 4

Since $q > 0$, $q = 4$

c Gradient of AB $= \frac{2-0}{-5-1} = -\frac{1}{3}$

Gradient of AC $= \frac{2-4}{-5-(-3)} = 1$

Gradient of BC $= \frac{0-4}{1-(-3)} = -1$

d $m_1 \times m_2 = -1$ so AC and BC are perpendicular.

Since the angle in a semicircle is a right angle, AB is a diameter of the circle.

5 a Gradient of $L_1 = \frac{6-1}{5-(-10)} = \frac{1}{3}$

Gradient of $L_2 = \frac{4-(-2)}{9-11} = -3$

$m_1 \times m_2 = -1$ so L_1 and L_2 are perpendicular.

b Because L_1 and L_2 are perpendicular and intersect at point W on the circumference, SV is a diameter.

Centre $= \left(\frac{5+11}{2}, \frac{6-2}{2}\right) = (8, 2)$

For the radius:

$d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$

$r^2 = (5-8)^2 + (6-2)^2$

$r^2 = (-3)^2 + (4)^2 = 25$

The equation of C is given by

$(x-8)^2 + (y-2)^2 = 25$

Expanding and simplifying: $x^2 + y^2 - 16x - 4y + 43 = 0$

6 a $PR^2 = PQ^2 + QR^2$

$PR^2 = (1-(t+17))^2 + ((t+7)-3)^2$

$= (-16-t)^2 + (t+4)^2 = 2t^2 + 40t + 272$

$PQ^2 = (1-3)^2 + ((t+7)-(t+11))^2 = (-2)^2 + (-4)^2 = 20$

$QR^2 = (3-(t+17))^2 + ((t+11)-3)^2$

$= (-14-t)^2 + (t+8)^2 = 2t^2 + 44t + 260$

Hence $2t^2 + 40t + 272 = 2t^2 + 44t + 260 + 20$

$-8 = 4t$

$t = -2$

Hence the coordinates of P and R are (1, 5) and (15, 3).

Centre of circle $= \left(\frac{1+15}{2}, \frac{5+3}{2}\right) = (8, 4)$

For the radius:

$d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$

$r^2 = (8-1)^2 + (4-5)^2$

$r^2 = (7)^2 + (-1)^2 = 50$

The equation of C is given by $(x-8)^2 + (y-4)^2 = 50$.

b $JL^2 = JK^2 + KL^2$

$JL^2 = (-11-7)^2 + (6-22)^2 = (-18)^2 + (-16)^2 = 580$

$JK^2 = (-11-(-1))^2 + (6-(p-8))^2$

$= (-10)^2 + (14-p)^2 = p^2 - 28p + 296$

$KL^2 = (-1-7)^2 + ((p-8)-22)^2$

$= (-8)^2 + (p-30)^2 = p^2 - 60p + 964$

Hence $580 = p^2 - 28p + 296 + p^2 - 60p + 964$

$0 = 2p^2 - 88p + 680$

$0 = p^2 - 44p + 340$

$0 = (p-10)(p-34)$

$p = 10$ or 34

Exercise 5.3A

p 142

1 a $\left(\frac{9+7}{2}, \frac{5-3}{2}\right) = (8, 1)$

b Gradient of DE $= \frac{5-(-3)}{9-7} = 4$

c Gradient of FG $= -\frac{1}{4}$

d Equation is given by $y - 1 = -\frac{1}{4}(x-8)$.

$4y + x - 12 = 0$.

e When $x = 4$, $y - 1 = -\frac{1}{4}(4-8)$

$y = 1 - 1 + 2 = 2$ as required.

f For the radius:

$d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$

$r^2 = (9-4)^2 + (5-2)^2$

$r^2 = (5)^2 + (3)^2 = 34$

The equation is given by $(x-4)^2 + (y-2)^2 = 34$.

2 Gradient of JK $= \frac{7-2}{3-10} = -\frac{5}{7}$

Gradient of bisector $= \frac{7}{5}$

Midpoint $= \left(\frac{3+10}{2}, \frac{7+2}{2}\right) = \left(\frac{13}{2}, \frac{9}{2}\right)$

Equation of bisector is given by $y - \frac{9}{2} = \frac{7}{5}\left(x - \frac{13}{2}\right)$

When $y = 8$, $8 - \frac{9}{2} = \frac{7}{5}\left(x - \frac{13}{2}\right)$

$\frac{7}{2} = \frac{7}{5}x - \frac{91}{10}$

$\frac{63}{5} = \frac{7}{5}x$

$x = 9$

Centre of circle = (9, 8)

For the radius:

$d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$

$r^2 = (9-3)^2 + (8-7)^2$

$r^2 = (6)^2 + (1)^2 = 37$

The equation is given by $(x-9)^2 + (y-8)^2 = 37$.

3 a Midpoint $= \left(\frac{10+x}{2}, \frac{11+y}{2}\right) = (6, 3)$

$10 + x = 2 \times 6$

$x = 2$

$11 + y = 2 \times 3$

$y = -5$

Coordinates of V are (2, –5).

b Gradient of UV $= \frac{11-3}{10-6} = 2$

Gradient of bisector $= -\frac{1}{2}$

Equation of bisector is given by $y - 3 = -\frac{1}{2}(x - 6)$.

When $x = 8$, $y - 3 = -\frac{1}{2}(8 - 6)$

$y = 2$

Centre of circle = (8, 2)

For the radius:

$d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$

$r^2 = (8 - 10)^2 + (2 - 11)^2$

$r^2 = (-2)^2 + (-9)^2 = 85$

The equation is given by $(x - 8)^2 + (y - 2)^2 = 85$.

4 $d^2 = 17^2 - 15^2 = 64$

$d = \sqrt{64} = 8$

$2 \times 8 + 2 \times 6 = 28\,\text{m}$

5 a Equation of circle can be rewritten as $(x + 3)^2 + (y + 5)^2 = 185$

Hence centre of circle = (–3, –5)

b When $y = 6$, $(x + 3)^2 + (6 + 5)^2 = 185$

$(x + 3)^2 + 121 = 185$

$(x + 3)^2 = 64$

$x + 3 = \pm 8$

$x = 5$ or -11

Length of chord = 5 – (–11) = 16

c Height of triangle = 6 – (–5) = 11

Base of triangle = PQ = 16

Area of triangle $= \frac{1}{2}bh = \frac{1}{2} \times 16 \times 11 = 88$

6 a Midpoint of AB $= \left(\frac{-4+2}{2}, \frac{3+7}{2}\right) = (-1, 5)$

Gradient of AB $= \frac{7-3}{2-(-4)} = \frac{2}{3}$

Gradient of bisector $= -\frac{3}{2}$

Equation of bisector is given by $y - 5 = -\frac{3}{2}(x + 1)$

$2y - 10 = -3x - 3$

$3x + 2y = 7$

b Midpoint of BC $= \left(\frac{2+10}{2}, \frac{7-5}{2}\right) = (6, 1)$

Gradient of BC $= \frac{7-(-5)}{2-10} = -\frac{3}{2}$

Gradient of bisector $= \frac{2}{3}$

Equation of bisector is given by $y - 1 = \frac{2}{3}(x - 6)$.

$3y - 3 = 2x - 12$

$2x = 3y + 9$

c 90° because gradients are perpendicular.

d Gradient AB $= \frac{2}{3}$ from **a**

Gradient BC $= -\frac{3}{2}$ from **b**

Hence AB is perpendicular to BC and angle ABC = 90˚.

e AC is a diameter.

7 For each circle, let the first point = A, the second = B and the third = C.

a A(12, 8), B(11, 1) and C(20, 4)

Midpoint of AB $= \left(\frac{12+11}{2}, \frac{8+1}{2}\right) = \left(\frac{23}{2}, \frac{9}{2}\right)$

Gradient of AB $= \frac{8-1}{12-11} = 7$

Gradient of bisector $= -\frac{1}{7}$

Equation of bisector is given by

$y - \frac{9}{2} = -\frac{1}{7}\left(x - \frac{23}{2}\right)$

$y = -\frac{1}{7}x + \frac{43}{7}$

Midpoint of AC $= \left(\frac{12+20}{2}, \frac{8+4}{2}\right) = (16, 6)$

Gradient of AC $= \frac{8-4}{12-20} = -\frac{1}{2}$

Gradient of bisector = 2

Equation of bisector is given by $y - 6 = 2(x - 16)$

$y = 2x - 26$

For point of intersection, $-\frac{1}{7}x + \frac{43}{7} = 2x - 26$

$\frac{225}{7} = \frac{15}{7}x$

$x = 15$

When $x = 15$, $y = 2 \times 15 - 26 = 4$

Centre of circle = (15, 4)

For the radius:

$d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$

$r^2 = (15 - 12)^2 + (4 - 8)^2$

$r^2 = (3)^2 + (-4)^2 = 25$

The equation is given by $(x - 15)^2 + (y - 4)^2 = 25$.

b A(3, 8), B(1, 4) and C(–6, 5)

Midpoint of AB $= \left(\frac{3+1}{2}, \frac{8+4}{2}\right) = (2, 6)$

Gradient of AB $= \frac{8-4}{3-1} = 2$

Gradient of bisector $= -\frac{1}{2}$

Equation of bisector is given by $y - 6 = -\frac{1}{2}(x - 2)$

$y = -\frac{1}{2}x + 7$

Midpoint of AC $= \left(\frac{3-6}{2}, \frac{8+5}{2}\right) = \left(-\frac{3}{2}, \frac{13}{2}\right)$

Gradient of AC $= \frac{8-5}{3-(-6)} = \frac{1}{3}$

Gradient of bisector = –3

Equation of bisector is given by
$y - \frac{13}{2} = -3(x - (-\frac{3}{2}))$
$y = -3x + 2$
For point of intersection, $-\frac{1}{2}x + 7 = -3x + 2$
$\frac{5}{2}x = -5$
$x = -2$
When $x = -2$, $y = -3 \times -2 + 2 = 8$
Centre of circle = (–2, 8)
For the radius:
$d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$
$r^2 = (-2 - 1)^2 + (8 - 4)^2$
$r^2 = (-3)^2 + (4)^2 = 25$
The equation is given by $(x + 2)^2 + (y - 8)^2 = 25$.

c A(19, 4), B(17, 0) and C(4, –1)
Midpoint of AB $= \left(\frac{19+17}{2}, \frac{4+0}{2}\right) = (18, 2)$
Gradient of AB $= \frac{4-0}{19-17} = 2$
Gradient of bisector $= -\frac{1}{2}$
Equation of bisector is given by $y - 2 = -\frac{1}{2}(x - 18)$
$y = -\frac{1}{2}x + 11$
Midpoint of AC $= \left(\frac{19+4}{2}, \frac{4-1}{2}\right) = \left(\frac{23}{2}, \frac{3}{2}\right)$
Gradient of AC $= \frac{4-(-1)}{19-4} = \frac{1}{3}$
Gradient of bisector $= -3$
Equation of bisector is given by $y - \frac{3}{2} = -3(x - \frac{23}{2})$
$y = -3x + 36$
For point of intersection, $-\frac{1}{2}x + 11 = -3x + 36$
$\frac{5}{2}x = 25$
$x = 10$
When $x = 10$, $y = -3 \times 10 + 36 = 6$
Centre of circle = (10, 6)
For the radius:
$d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$
$r^2 = (10 - 17)^2 + (6 - 0)^2$
$r^2 = (-7)^2 + (6)^2 = 85$
The equation is given by $(x - 10)^2 + (y - 6)^2 = 85$.

d A(–5, –23), B(–17, –19) and C(–1, –3)
Midpoint of AB $= \left(\frac{-5-17}{2}, \frac{-23-19}{2}\right) = (-11, -21)$
Gradient of AB $= \frac{-19-(-23)}{-17-(-5)} = -\frac{1}{3}$
Gradient of bisector = 3
Equation of bisector is given by $y + 21 = 3(x + 11)$
$y = 3x + 12$
Midpoint of AC $= \left(\frac{-5-1}{2}, \frac{-23-3}{2}\right) = (-3, -13)$
Gradient of AC $= \frac{-3-(-23)}{-1-(-5)} = 5$
Gradient of bisector $= -\frac{1}{5}$
Equation of bisector is given by $y + 13 = -\frac{1}{5}(x + 3)$
$y = -\frac{1}{5}x - \frac{68}{5}$
For point of intersection, $3x + 12 = -\frac{1}{5}x - \frac{68}{5}$
$\frac{16}{5}x = -\frac{128}{5}$
$x = -8$
When $x = -8$, $y = 3 \times -8 + 12 = -12$
Centre of circle = (–8, –12)
For the radius:
$d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$
$r^2 = (-8 + 1)^2 + (-12 + 3)^2$
$r^2 = (-7)^2 + (-9)^2 = 130$
The equation is given by $(x + 8)^2 + (y + 12)^2 = 130$.

8 a Circle A: $x^2 + y^2 + 12x - 12y = 73$ ①
Circle B: $x^2 + y^2 - 16x + 2y = -25$ ②
Subtract ② from ①.
$28x - 14y = 98$
$2x - y = 7$
Hence the line on which the circles intersect is given by $y = 2x - 7$.
Substitute $y = 2x - 7$ into the equation for Circle A.
$x^2 + (2x - 7)^2 + 12x - 12(2x - 7) = 73$
$5x^2 - 40x + 60 = 0$
$x^2 - 8x + 12 = 0$
$(x - 2)(x - 6) = 0$
$x = 2$ or 6
When $x = 2$, $y = 2 \times 2 - 7 = -3$
When $x = 6$, $y = 2 \times 6 - 7 = 5$
Points of intersection are (2, –3) and (6, 5).

b Length of chord:
$d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$
$d^2 = (6 - 2)^2 + (5 - (-3))^2$
$d^2 = (4)^2 + (8)^2 = 80$
The length of the chord is $\sqrt{80}$ is $= 4\sqrt{5}$

Exercise 5.4A p 146

1 Gradient of radius $= \frac{6-8}{4-7} = \frac{2}{3}$
Gradient of the tangent $= -\frac{3}{2}$
The equation of the tangent is given by
$y - 8 = -\frac{3}{2}(x - 7)$
$2y - 16 = -3x + 21$
$3x + 2y = 37$

2 a $2^2 + (-2 - 1)^2 = 4 + 9 = 13$
Since both sides of the equation agree, T lies on the circle.

b Centre of circle = (0, 1)

Gradient of radius = $\frac{1-(-2)}{0-2} = -\frac{3}{2}$

Gradient of the tangent = $\frac{2}{3}$

The equation of the tangent is given by

$y + 2 = \frac{2}{3}(x - 2)$

$3y + 6 = 2x - 4$

Hence $2x = 3y + 10$

3 **a** Centre = (3, 6)

b Radius = 8

c Let centre = X, point P = (15, 7) and point where tangent meets circle = T.

The length of PT is required, where

$PT^2 + XT^2 = PX^2$

For PX:

$$d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$$

$$PX^2 = (3 - 15)^2 + (6 - 7)^2$$

$$PX^2 = (-12)^2 + (-1)^2 = 145$$

$PT^2 + XT^2 = PX^2$

$PT^2 + 8^2 = 145$

$PT^2 = 81$

$PT = \sqrt{81} = 9\,\text{m}$

4 For A:

Gradient of radius = $\frac{11-7}{7-4} = \frac{4}{3}$

Gradient of the tangent = $-\frac{3}{4}$

The equation of the tangent is given by

$y - 7 = -\frac{3}{4}(x - 4)$

Hence $y = -\frac{3}{4}(x - 4) + 7$

For B:

Gradient of radius = $\frac{11-7}{7-10} = -\frac{4}{3}$

Gradient of the tangent = $\frac{3}{4}$

The equation of the tangent is given by

$y - 7 = \frac{3}{4}(x - 10)$

Hence $y = \frac{3}{4}(x - 10) + 7$

Put tangents equal.

$-\frac{3}{4}(x - 4) + 7 = \frac{3}{4}(x - 10) + 7$

$-\frac{3}{4}(x - 4) = \frac{3}{4}(x - 10)$

$-(x - 4) = x - 10$

$-x + 4 = x - 10$

$14 = 2x$

$x = 7$ (this could also be deduced by symmetry since AB is horizontal)

When $x = 7$, $y = \frac{3}{4}(7 - 10) + 7$

$y = \frac{3}{4}(-3) + 7$

$y = \frac{19}{4}$

Hence the coordinates of Q are $(7, \frac{19}{4})$.

b AXBQ is a kite. Split into congruent triangles AXQ and BXQ.

Base of triangle AXQ = $11 - \frac{19}{4} = \frac{25}{4}$

Perpendicular height of triangle AXQ = 3

Area of triangle AXQ = $\frac{1}{2} \times \frac{25}{4} \times 3 = \frac{75}{8}$

Area of AXBQ = $2 \times \frac{75}{8} = \frac{75}{4}$

5 Rewrite the equation of the circle as

$(x + 4)^2 + (y - 1)^2 = 81$

Centre = (–4, 1), radius = 9

Let centre = X and point where tangent meets circle = T.

$PT^2 + XT^2 = PX^2$

For PX:

$$d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$$

$$PX^2 = (7 - (-4))^2 + (4 - 1)^2$$

$$PX^2 = (11)^2 + (3)^2 = 130$$

$PT^2 + XT^2 = PX^2$

$PT^2 + 9^2 = 130$

$PT^2 = 49$

$PT = \sqrt{49} = 7$

6 **a** Rewrite the equation of the circle as

$(x - 15)^2 + (y - 23)^2 = 400$

Centre = (15, 23), radius = 20

Gradient of radius = $\frac{35-23}{31-15} = \frac{3}{4}$

Gradient of the tangent = $-\frac{4}{3}$

The equation of the tangent is given by

$y - 35 = -\frac{4}{3}(x - 31)$

$3y - 105 = -4x + 124$

$4x + 3y = 229$

When $x = 10$, $y = q$

$40 + 3q = 229$

$3q = 189$

$q = 63$

b Use $d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$

For PT:

$PT^2 = (10 - 31)^2 + (63 - 35)^2$

$PT^2 = (-21)^2 + (28)^2 = 1225$

For PX:

$PX^2 = (10 - 15)^2 + (63 - 23)^2$

$PX^2 = (-5)^2 + (40)^2 = 1625$

TX is the radius, so $TX^2 = 400$

Hence $PT^2 + TX^2 = PX^2$ because $1225 + 400 = 1625$.

c Since triangle is right-angled,
area $= \frac{1}{2} \times TX \times PT$.

$TX = 20$

$PT = \sqrt{1225} = 35$

Area $= \frac{1}{2} \times 20 \times 35 = 350$

7 a Gradient of tangent $= 1$

Gradient of radius $= -1$

Equation of line with gradient -1 through $(5, 11)$ is given by $y - 11 = -(x - 5)$

$$y = 16 - x$$

Find point of intersection of $y = 16 - x$ and $y = x + 2$.

$16 - x = x + 2$

$14 = 2x$

$x = 7$

$y = 7 + 2 = 9$

Hence the point $(7, 9)$ lies on the circumference.

For the radius:

$d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$

$r^2 = (7 - 5)^2 + (9 - 11)^2$

$r^2 = (2)^2 + (-2)^2 = 8$

The equation of the circle is given by $(x - 5)^2 + (y - 11)^2 = 8$

b Gradient of tangent $= 3$

Gradient of radius $= -\frac{1}{3}$

Equation of line with gradient $-\frac{1}{3}$ through $(-8, 23)$ is given by $y - 23 = -\frac{1}{3}(x + 8)$

$$y = -\frac{1}{3}(x + 8) + 23$$

Find point of intersection of $y = -\frac{1}{3}(x + 8) + 23$ and $y = 3x + 5$.

$-\frac{1}{3}(x + 8) + 23 = 3x + 5$

$-(x + 8) + 69 = 9x + 15$

$-x - 8 + 69 = 9x + 15$

$46 = 10x$

$x = 4.6$

$y = 3 \times 4.6 + 5 = 18.8$

Hence the point $(4.6, 18.8)$ lies on the circumference.

For the radius:

$d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$

$r^2 = (-8 - 4.6)^2 + (23 - 18.8)^2$

$r^2 = (-12.6)^2 + (4.2)^2 = 176.4$

The equation of the circle is given by $(x + 8)^2 + (y - 23)^2 = 176.4$

Exercise 5.4B — p 150

1 a Substitute $y = x - 10$ into $x^2 + y^2 = 50$.

$x^2 + (x - 10)^2 = 50$

$x^2 + x^2 - 20x + 100 = 50$

$2x^2 - 20x + 50 = 0$

$x^2 - 10x + 25 = 0$

$(x - 5)^2 = 0$

The equation has repeated roots so $y = x - 10$ is a tangent of the circle.

b Substitute $x = 7y - 50$ into $x^2 + y^2 = 50$.

$(7y - 50)^2 + y^2 = 50$

$49y^2 - 700y + 2500 + y^2 = 50$

$50y^2 - 700y + 2450 = 0$

$y^2 - 14y + 49 = 0$

$(y - 7)^2 = 0$

The equation has repeated roots so $7y = x + 50$ is a tangent of the circle.

2 Substitute $y = 4x - 5$ into $x^2 + y^2 + 10x - 18y + 38 = 0$.

$x^2 + (4x - 5)^2 + 10x - 18(4x - 5) + 38 = 0$

$x^2 + 16x^2 - 40x + 25 + 10x - 72x + 90 + 38 = 0$

$17x^2 - 102x + 153 = 0$

$x^2 - 6x + 9 = 0$

$(x - 3)^2 = 0$

The equation has repeated roots so $y = 4x - 5$ is a tangent of the circle.

3 Substitute $y = 3x + c$ into $x^2 + y^2 - 10x - 6y - 6 = 0$.

$x^2 + (3x + c)^2 - 10x - 6(3x + c) - 6 = 0$

$x^2 + 9x^2 + 6cx + c^2 - 10x - 18x - 6c - 6 = 0$

$10x^2 + (6c - 28)x + (c^2 - 6c - 6) = 0$

For $b^2 - 4ac = 0$:

$(6c - 28)^2 - 4 \times 10 \times (c^2 - 6c - 6) = 0$

$36c^2 - 336c + 784 - 40c^2 + 240c + 240 = 0$

$-4c^2 - 96c + 1024 = 0$

$c^2 + 24c - 256 = 0$

$(c - 8)(c + 32) = 0$

$c = 8$ or -32

4 Substitute $y = -\frac{1}{2}x + c$ into $x^2 + y^2 - 12x - 26y + 125 = 0$.

$x^2 + (-\frac{1}{2}x + c)^2 - 12x - 26(-\frac{1}{2}x + c) + 125 = 0$

$x^2 + \frac{1}{4}x^2 - cx + c^2 - 12x + 13x - 26c + 125 = 0$

$\frac{5}{4}x^2 + (1 - c)x + (c^2 - 26c + 125) = 0$

$5x^2 + (4 - 4c)x + (4c^2 - 104c + 500) = 0$

For $b^2 - 4ac = 0$:

$$(4 - 4c)^2 - 4 \times 5 \times (4c^2 - 104c + 500) = 0$$
$$16 - 32c + 16c^2 - 80c^2 + 2080c - 10\,000 = 0$$
$$-64c^2 + 2048c - 9984 = 0$$
$$c^2 - 32c + 156 = 0$$
$$(c - 6)(c - 26) = 0$$
$$c = 6 \text{ or } 26$$

The tangents are $y = -\frac{1}{2}x + 6$ and $y = -\frac{1}{2}x + 26$.

5 Substitute $y = mx + 24$ into $(x - 18)^2 + (y - 10)^2 = 52$.

$$(x - 18)^2 + (mx + 24 - 10)^2 = 52$$
$$(x - 18)^2 + (mx + 14)^2 = 52$$
$$x^2 - 36x + 324 + m^2x^2 + 28mx + 196 = 52$$
$$(1 + m^2)x^2 + (28m - 36)x + 468 = 0$$

For $b^2 - 4ac = 0$:

$$(28m - 36)^2 - 4 \times (1 + m^2) \times 468 = 0$$
$$784m^2 - 2016m + 1296 - 1872 - 1872m^2 = 0$$
$$-1088m^2 - 2016m - 576 = 0$$
$$34m^2 + 63m + 18 = 0$$
$$(17m + 6)(2m + 3) = 0$$
$$m = -\frac{6}{17} \text{ or } -\frac{3}{2}$$

For the tangent $y = -\frac{6}{17}x + 24$:

$$17y = -6x + 408$$
$$6x + 17y - 408 = 0$$

For the tangent $y = -\frac{3}{2}x + 24$:

$$2y = -3x + 48$$
$$3x + 2y - 48 = 0$$

Exam-style questions 5 p 151

1 **a** The equation of the circle is given by $(x - a)^2 + (y - b)^2 = r^2$, so centre = $(9, -8)$. Gemma has copied the signs from the equation but needed to change them both.

b Olivia has stated the length of the radius, not the diameter.

Radius = $\sqrt{484} = 22$

Diameter = $22 \times 2 = 44$

2 Radius = 8

Equation is given by $(x - 7)^2 + (y + 3)^2 = 64$

Expand.

$$x^2 - 14x + 49 + y^2 + 6y + 9 = 64$$
$$x^2 + y^2 - 14x + 6y - 6 = 0$$

3 On the x-axis, $y = 0$.

$$(x + 5)^2 + (-7)^2 = 50$$
$$(x + 5)^2 = 1$$
$$x + 5 = \pm 1$$
$$x = -5 \pm 1 = -6 \text{ or } -4$$

Coordinates are $(-6, 0)$ and $(-4, 0)$.

On the y-axis, $x = 0$

$$(x + 5)^2 + (y - 7)^2 = 50$$
$$(5)^2 + (y - 7)^2 = 50$$
$$(y - 7)^2 = 25$$
$$y - 7 = \pm 5$$
$$y = 7 \pm 5 = 2 \text{ or } 12$$

Coordinates are $(0, 2)$ and $(0, 12)$.

4 Dilgusha has not halved the coefficients of x and y when completing the square.

She has added 14^2 rather than subtracting.

When square-rooting -182, which shouldn't have been possible in this situation, she has square-rooted $+182$ and made it negative.

Correct solution:

Complete the square on x and y.

$$(x + 3)^2 - 9 + (y - 7)^2 - 49 + 22 = 0$$
$$(x + 3)^2 + (y - 7)^2 = 36$$

Centre = $(-3, 7)$

Radius = $\sqrt{36} = 6$

5 **a** $d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$

$r^2 = (180 - 60)^2 + (40 - 130)^2$

$r^2 = (120)^2 + (-90)^2 = 22\,500$

Radius = $\sqrt{22\,500} = 150$ cm

b The circle has the equation $(x - 60)^2 + (y - 130)^2 = 22\,500$

c Gradient of radius = $\frac{130 - 40}{60 - 180} = -\frac{3}{4}$

Gradient of the tangent = $\frac{4}{3}$

The equation of the tangent is given by

$y - 40 = \frac{4}{3}(x - 180)$

$3y - 120 = 4x - 720$

$3y - 4x + 600 = 0$

d $d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$

$d^2 = (5940 - 180)^2 + (4360 - 40)^2$

$d^2 = (5760)^2 + (4320)^2 = 51\,840\,000$

Distance = $\sqrt{51\,840\,000} = 7200$ cm = 72 m

e Unlikely athlete will remain at same point whilst turning.

Hammer's motion will not be horizontal because it will be affected by gravity.

6 **a** Midpoint of AB $= \left(\frac{9+13}{2}, \frac{10-2}{2}\right) = (11, 4)$

For the radius, $d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$

$r^2 = (11-9)^2 + (4-10)^2$

$r^2 = (2)^2 + (-6)^2 = 40$

The circle has the equation $(x-11)^2 + (y-4)^2 = 40$.

b Gradient of radius $= \frac{10-4}{9-11} = -3$

Gradient of the tangent $= \frac{1}{3}$

The equation of the tangent is given by

$y - 10 = \frac{1}{3}(x-9)$

$3y - 30 = x - 9$

$x - 3y + 21 = 0$

7 Centre of circle, X = (–9, 4)

Radius $= \sqrt{49} = 7$

$PX^2 = (8-(-9))^2 + (21-4)^2$

$PX^2 = (17)^2 + (17)^2 = 578$

$PT^2 = PX^2 - XT^2$

$PT^2 = 578 - 7^2 = 529$

$PT = \sqrt{529} = 23$

8 **a** $r^2 = (31-7)^2 + (12-2)^2$

$r^2 = (24)^2 + (10)^2 = 676$

Equation is given by $(x-7)^2 + (y-2)^2 = 676$.

b Midpoint of XA $= \left(\frac{7+31}{2}, \frac{2+12}{2}\right) = (19, 7)$

Gradient of XA $= \frac{12-2}{31-7} = \frac{5}{12}$

Gradient of the tangent $= -\frac{12}{5}$

The equation of the tangent is given by

$y - 7 = -\frac{12}{5}(x-19)$ or $12x + 5y - 263 = 0$.

c Let P be the midpoint of XA.

XM = XA = AM because XM and XA are radii and since MN is the bisector of XA it is a line of symmetry for triangle MAX (hence XM = AM).

$MP^2 = MX^2 - XP^2$

$MP^2 = 26^2 - 13^2 = 507$

$MP = \sqrt{507} = 13\sqrt{3}$

$MN = 2 \times 13\sqrt{3} = 26\sqrt{3}$

9 **a** $(x+4)^2 + (y-11)^2 = 50$

b $y = 7 - x$

$(x+4)^2 + (7-x-11)^2 = 50$

$(x+4)^2 + (-4-x)^2 = 50$

$x^2 + 8x + 16 + x^2 + 8x + 16 = 50$

$2x^2 + 16x - 18 = 0$

$x^2 + 8x - 9 = 0$

$(x+9)(x-1) = 0$

$x = -9$ or 1

When $x = -9$, $y = 7 - (-9) = 16$

When $x = 1$, $y = 7 - 1 = 6$

Since the x-coordinate of A is less than the x-coordinate of B, A is (–9, 16) and B is (1, 6).

c Tangent at A:

Gradient of radius $= \frac{16-11}{-9-(-4)} = -1$

Gradient of the tangent = 1

(Note that since the centre of the circle lies on $x + y = 7$, AB is a diameter, so the tangent is perpendicular to $x + y = 7$.)

The equation of the tangent is given by $y - 16 = 1(x+9)$

Hence $y = x + 25$

Tangent at P:

Gradient of radius $= \frac{18-11}{-3-(-4)} = 7$

Gradient of the tangent $= -\frac{1}{7}$

The equation of the tangent is given by

$y - 18 = -\frac{1}{7}(x+3)$

Hence $y = -\frac{1}{7}(x+3) + 18$

From which, $x + 25 = -\frac{1}{7}(x+3) + 18$

$x + 25 = -\frac{1}{7}x - \frac{3}{7} + 18$

$7x + 175 = -x - 3 + 126$

$8x = -52$

$x = -\frac{13}{2}$

When $x = -\frac{13}{2}$, $y = -\frac{13}{2} + 25 = \frac{37}{2}$

Coordinates of T are $(-\frac{13}{2}, \frac{37}{2})$.

10 **a** Equation is given by $(x-6)^2 + (y-3)^2 = 9$

The centre is (6, 3). The radius is 3.

Method 1:

Similar triangles PTX and PAO:

$\frac{XP}{OP} = \frac{TX}{OA}$

Let length of XP = a

$\frac{a}{\sqrt{6^2 + (3+a)^2}} = \frac{3}{6}$

$2a = \sqrt{36 + (3+a)^2}$

$4a^2 = 36 + (3+a)^2$

$4a^2 = 36 + 9 + 6a + a^2$

$3a^2 - 6a - 45 = 0$

$a^2 - 2a - 15 = 0$

$(a-5)(a+3) = 0$

$a = 5$ (can't be –3)

Method 2:

Since $y = mx$ at T, $(x-6)^2 + (mx-3)^2 = 9$

$x^2 - 12x + 36 + m^2x^2 - 6mx + 9 = 9$

$(1+m^2)x^2 + (-12-6m)x + 36 = 0$

Since $y = mx$ is a tangent, $b^2 - 4ac = 0$.

$$(-12 - 6m)^2 - 4 \times (1 + m^2) \times 36 = 0$$
$$144 + 144m + 36m^2 - 144 - 144m^2 = 0$$
$$4 + 4m + m^2 - 4 - 4m^2 = 0$$
$$3m^2 - 4m = 0$$
$$m(3m - 4) = 0$$
$$m = 0 \text{ or } \tfrac{4}{3}$$

Since $m = \frac{4}{3}$, then $\frac{AP}{OA} = \frac{4}{3}$

$$\frac{3 + XP}{6} = \frac{4}{3}$$
$$3(3 + XP) = 24$$
$$3 + XP = 8$$
$$XP = 5$$

b Quadrilateral OAXT is made from two congruent right-angled triangles, each with a base of 6 and a height of 3.

Area of triangle OAX $= \frac{1}{2} \times 6 \times 3 = 9$

Area of OAXT $= 2 \times 9 = 18$

11 a Substitute $y = mx$ into the equation of the circle C.

$$(x - 5)^2 + (mx - 3)^2 = 2$$
$$x^2 - 10x + 25 + m^2x^2 - 6mx + 9 = 2$$
$$(1 + m^2)x^2 + (-10 - 6m)x + 32 = 0$$

Given that $y = mx$ is a tangent to the circle C, $b^2 - 4ac = 0$:

$$(-10 - 6m)^2 - 4 \times (1 + m^2) \times 32 = 0$$
$$100 + 120m + 36m^2 - 128 - 128m^2 = 0$$
$$92m^2 - 120m + 28 = 0$$
$$23m^2 - 30m + 7 = 0$$

b $(23m - 7)(m - 1) = 0$

$m = 1$ or $\frac{7}{23}$

Both tangents are the same length, so can use either value for m. Use 1.

When $m = 1$, the tangent equation is given by $y = x$.

Substitute $y = x$ into the circle equation.

$$(x - 5)^2 + (x - 3)^2 = 2$$
$$2x^2 - 16x + 32 = 0$$
$$x^2 - 8x + 16 = 0$$
$$(x - 4)^2 = 0$$
$$x = 4$$
$$y = 4$$

Length of tangent $= \sqrt{4^2 + 4^2} = 4\sqrt{2}$

12 130 units^2

6 Trigonometry

Prior knowledge — p 155

1 $30 \sin 35°$ (or $30 \cos 55°$) $= 17.2$ cm

2 $\tan C = \frac{8}{6} = 1.33$

$C = \tan^{-1} 1.33 = 53.1°$

Alternatively, $\sin C = 0.8$ or $\cos C = 0.6$

3 Factorise: $(2x - 1)(x + 1) = 0$

Either $2x - 1 = 0 \Rightarrow x = \frac{1}{2}$

or $x + 1 = 0 \Rightarrow x = -1$

$x = \frac{1}{2}$ or -1

Exercise 6.1A — p 157

1 $180 - 80 = 100°$

2 **a** $180 - 12 = 168°$ **b** $180 - 24 = 156°$

c $180 - 36 = 144°$ **d** $180 - 48 = 132°$

e $180 - 72 = 108°$

3 $180 - 37 = 143$ so $\cos 143° = -0.7986$

4 $180 - 81 = 99$ and $\cos 99° = -\cos 81°$ so $\theta = 99°$

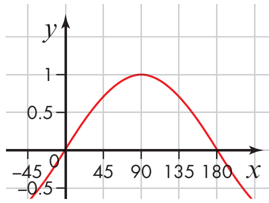
5 **a** $\cos^{-1} 0.33 = 70.7°$ and this is the only solution in the given range.

b $\cos^{-1} (-0.33) = 109.3°$ and this is the only solution in the given range.

6 **a**

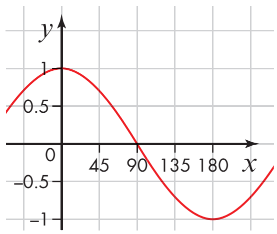
angle	0	30°	60°	90°	120°	150°	180°
sine	0	0.5	0.866	1	0.866	0.5	0
cosine	1	0.866	0.5	0	–0.5	–0.866	–1

b



c Reflection symmetry in the line $x = 90°$

d



e Rotation symmetry of order 2, centre (90°, 0)

f 45°

g 135°

7 a $\sin^{-1} 0.66 = 41.3°$ and this is one solution.

Another possible value is 180 – 41.3 = 138.7°

b There is no solution. No angle in the given range has a negative sine.

Exercise 6.2A p 164

1 a $BC^2 = 20^2 + 30^2 - 2 \times 20 \times 30 \times \cos 65°$

$\Rightarrow BC = 28.2\,\text{cm}$

b Area $= \frac{1}{2} \times 20 \times 30 \times \sin 65° = 271.9\,\text{cm}^2$

2 a Area $= \frac{1}{2} \times 35 \times 40 \times \sin 70° = 658\,\text{cm}^2$

b $\frac{BC}{\sin 70°} = \frac{40}{\sin 60°}$

$\Rightarrow BC = \frac{40 \sin 70°}{\sin 60°} = 43.4\,\text{cm}$

3 $\cos X = \frac{10^2 + 11^2 - 12^2}{2 \times 10 \times 11} \Rightarrow X = 69.5°$

$\cos Y = \frac{10^2 + 12^2 - 11^2}{2 \times 10 \times 12} \Rightarrow Y = 59.2°$

$Z = 180 - (69.5 + 59.2) = 51.3°$

4 a $\frac{YZ}{\sin 42°} = \frac{40}{\sin 115°} \Rightarrow YZ = \frac{40 \sin 42°}{\sin 115°} = 29.5\,\text{cm}$

b Angle $Y = 180 - (42 + 115) = 23°$

$\frac{XZ}{\sin 23°} = \frac{40}{\sin 115°}$

$\Rightarrow XZ = \frac{40 \sin 23°}{\sin 115°} = 17.2\,\text{cm}$

5 $\frac{\sin DBC}{12} = \frac{\sin 45}{9} \Rightarrow \sin DBC = \frac{12 \sin 45}{9} = 0.9428$

Angle DBC could be 70.5° or 180 – 70.5 = 109.5°

6 $AC^2 = 25^2 + 21^2 - 2 \times 25 \times 21 \cos 60° = 541$

$\cos D = \frac{16^2 + 18^2 - AC^2}{2 \times 16 \times 18} = \frac{16^2 + 18^2 - 541}{2 \times 16 \times 18} \Rightarrow D = 86.1°$

7 $\cos BAC = \frac{35^2 + 44^2 - 29^2}{2 \times 35 \times 44} \Rightarrow$ angle BAC = 41°.

The bearing of C is $\Rightarrow 100 + 41 = 141°$

8 a Angle ATB = 42 – 18 = 24°

$\frac{BT}{\sin 18°} = \frac{15}{\sin 24°} \Rightarrow BT = \frac{15 \sin 18°}{\sin 24°} = 11.40$

Height of tree = BT sin 42° = 11.40 × sin 42° = 7.63 m

9 a $\frac{\sin Y}{40} = \frac{\sin 30°}{YZ} \Rightarrow \sin Y = \frac{40 \sin 30°}{YZ} = \frac{20}{YZ}$

i If YZ = 32 cm, $\sin Y = \frac{20}{32} = 0.625 \Rightarrow Y = 38.7°$ or 180 – 38.7 = 141.3°

There are two possible values for Y.

ii If YZ = 50 cm, $\sin Y = \frac{20}{50} = 0.4$

$\Rightarrow Y = 23.6°$ or 180 – 23.6 = 156.4°

The second of these is not a possible solution because 156.4 + 30 = 186.4° > 180°, the angle sum of a triangle. In this case, $Y = 23.6°$ is the only possible solution.

iii If YZ = 16 cm, $\sin Y = \frac{20}{16} = 1.25$

This equation has no solution because the sine of an angle cannot be greater than 1. YZ = 16 cm is impossible.

b $\sin Y = \frac{20}{YZ} \Rightarrow YZ = \frac{20}{\sin Y}$

YZ will have its smallest possible value when sin Y has its largest value. The largest value is 1 (when $Y = 90°$) so the smallest possible value of YZ is 20 cm.

Exercise 6.3A p 168

1 a 0.5 b –0.5 c –0.5

d –0.5 e 0.5

2 a –0.966 b 0.966 c 0.966

d –0.966 e 0.966

3 a 0.940 b 0.940 c 0.940

d 0.940 e –0.940

4 a 71.8° and 180 – 71.8 = 108.2°

b A calculator probably gives the value –20.5 which is out of the range $0° \leqslant x \leqslant 360°$.

The solutions are 180 + 20.5 = 200.5° and 360 – 20.5 = 339.5°.

c A calculator value is –54.3°.

Solutions are 180 + 54.3 = 234.3° and 360 – 54.3 = 305.7°.

d No solution because $-1 \leqslant \sin x \leqslant 1$.

5 a A calculator value is 104.5°.

The other solution is 360 − 104.5 = 255.5°.

b 84.3° and 360 – 84.3 = 275.7

c 90° and 360 − 90 = 270°

d 180° is the only solution.

6 a 77.3°, 360 − 77.3 = 282.7°, 360 + 77.3 = 437.3°, 720 − 77.3 = 642.7°

b 180 + 12.7 = 192.7°, 360 – 12.7 = 347.3°, 540 + 12.7 = 552.7°, 720 − 12.7 = 707.3°

c 0°, 180°, 360°, 540°, 720°

d 90°, 450°

7 a 0°, 360°, –360°

b –30°, –150°, 210°, 330°

c 45.6°, 314.4°, –45.6°, –314.6°

d 270°, –90°

8 Draw graphs of $y = \sin x$ and $y = \cos x$ on the same axes.

They cross where $x = 45°, 225°, -135°$ and $-315°$.

9 a Translation of $\begin{bmatrix}270\\0\end{bmatrix}$ or translation of $\begin{bmatrix}-450\\0\end{bmatrix}$ are two possible answers.

b Rotation of 180° about (135, 0) is one possible answer.

Exercise 6.4A p 171

1 $\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{-0.5}{0.866} = -0.577$

2 $\tan\theta = \frac{\sin\theta}{\cos\theta} \to -4.121 = \frac{\sin\theta}{-0.236}$
$\Rightarrow \sin\theta = -4.121 \times -0.236 = 0.973$

3 $305 - 180 = 125°$ and $305 - 360 = -55°$

4 a 2.9° and 180 + 2.9 = 182.9°

b 153.4° and 180 + 153.4 = 333.4°

c 78.7° and 180 + 78.7 = 258.7°

d A calculator answer is –88.9°.

Values in the specified range are
180 – 88.9 = 91.1° and 360 – 88.9 = 271.1°.

5 a 0, ±180°, ±360°, ±540°, etc.

b 45° ± any multiple of 180°

That is 45°, 225°, 405°, … and –135°, –315°, –495°, …

c –45° ± any multiple of 180°

That is 135°, 315°, 495°, … and –45°, –225°, –405°, …

6 $\frac{\sin x}{\cos x} = \tan x = 4 \Rightarrow x = 76.0°$ and $180 + 76.0 = 256.0°$

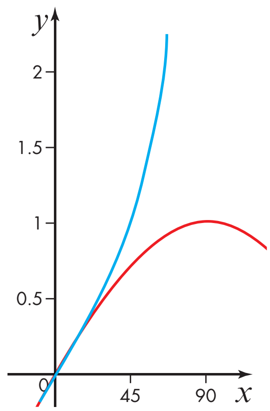
7 $5\sin x = -3\cos x \Rightarrow \frac{\sin x}{\cos x} = \frac{-3}{5} \Rightarrow \tan x = -0.6$
A calculator gives $x = -31.0°$ so the solutions are
180 – 31.0 = 149.0° and 360 – 31.0 = 329.0°.

8 a For example, a translation of $\begin{bmatrix}180\\0\end{bmatrix}$ or $\begin{bmatrix}360\\0\end{bmatrix}$ or $\begin{bmatrix}-180\\0\end{bmatrix}$

b For example, a rotation of 180° about (0, 0), (90, 0), (–90, 0) or (270, 0).

9 a $\tan\theta = \frac{\sin\theta}{\cos\theta}$. If $0° < \theta < 90°$ then $0 < \cos\theta < 1$.
Dividing $\sin\theta$ by a number less than 1 gives an answer that is greater than $\sin\theta$ so $\tan\theta > \sin\theta$.

b



$\tan x$ is always above $\sin x$.

Exercise 6.5A p 174

1 a $x = \sin^{-1} 0.835 = 56.6°$

b $2x = \sin^{-1} 0.835 \Rightarrow x = \frac{\sin^{-1} 0.835}{2} = 28.3°$

c $x = \sin^{-1} 0.835 - 10 \Rightarrow x = 46.6°$

d $x = \frac{1}{2}(\sin^{-1} 0.835 - 10) \Rightarrow x = 23.3°$

2 $2x = \cos^{-1} 0.5 = 60°, 300°, 420°, 660°, \ldots$
$\Rightarrow x = 30°, 150°, 210°, 330°$
There are four solutions in the given range.

3 $x - 30 = \sin^{-1} 0.8 = 53.1°, 126.9°, \ldots$
$\Rightarrow x = 83.1°$ or $156.9°$

4 $\sin 0.5x = \frac{5}{20} = 0.25 \Rightarrow 0.5x = 14.5°, 165.5°, \ldots$
$\Rightarrow x = 29.0°$ or $331.0°$

5 $2x = \tan^{-1} 1.5 = 56.3°, 236.3°, 416.3°, 596.3, \ldots$
$\Rightarrow x = 28.2°, 118.2°, 208.2°, 298.2°$

6 $\sin 10x = 0.9 \Rightarrow 10x = 64.2°, 115.8°, 424.2°, \ldots$
$\Rightarrow x = 6.4°$ and $11.6°$ are the two solutions.

7 a $2x + 60 = \sin^{-1} 0.7 = 44.4°, 135.6°, 404.4°, \ldots$
$\Rightarrow x = 37.8°$ or $172.2°$

8 a 7.2°

b $50x = \sin^{-1} 0.5 \Rightarrow 50x = 30°, 150°, 390°, \ldots$
$\Rightarrow x = -4.2°\ 0.6°$ or $3°$

c The number of cycles is 360 ÷ 7.2 = 50 and there are two solutions in each cycle. There will be 50 × 2 = 100 solutions.

Exercise 6.6A p 176

1 Calculator operation

2 a $\cos x = \pm\sqrt{1 - 0.68^2} = \pm 0.733$

b $\tan x = \frac{\sin x}{\cos x} = \pm\frac{0.68}{0.733} = \pm 0.927$

3 a $\sin x = \pm\sqrt{1 - 0.44^2} = \pm 0.898$

b $\tan x = \frac{\sin x}{\cos x} = \pm\frac{0.898}{0.44} = \pm 2.041$

4 One is a reflection of the other in the line $y = \frac{1}{2}$.
This ensures that corresponding values always add to 1.

5 $3\cos^2 x - 2\cos x = 0 \Rightarrow \cos x(3\cos x - 2) = 0$
Either $\cos x = 0 \Rightarrow x = 90°$ or $270°$
or $3\cos x - 2 = 0 \Rightarrow \cos x = \frac{2}{3} \Rightarrow x = 48.2°$ or $311.8°$
There are four solutions.

6 $\sin x + 1 = 1 - \sin^2 x \rightarrow \sin x + \sin^2 x = 0$
$\Rightarrow \sin x(1 + \sin x) = 0$
Either $\sin x = 0 \Rightarrow x = 0°, 180°, 360°$
or $1 + \sin x = 0 \Rightarrow \sin x = -1 \Rightarrow x = 270°$
There are four solutions.

7 $\cos^2 x = 0.25 \Rightarrow \cos x = \pm 0.5$
If $\cos x = 0.5$, $x = 60°$ or $300°$
If $\cos x = -0.5$, $x = 120°$ or $240°$

8 $(3\sin x - 1)(2\sin x - 1) = 0$
Either $3\sin x - 1 = 0 \Rightarrow \sin x = \frac{1}{3} \Rightarrow x = 19.5°, 160.5°$
or $2\sin x - 1 = 0 \Rightarrow \sin x = \frac{1}{2} \Rightarrow x = 30°, 150°$

9 $4\sin^2 x - 7\sin x - 2 = 0 \Rightarrow (4\sin x + 1)(\sin x - 2) = 0$
Either $4\sin x + 1 = 0 \Rightarrow \sin x = -\frac{1}{4}$
$\Rightarrow x = 194.5°, 345.5°$
or $\sin x - 2 = 0 \Rightarrow \sin x = 2$
$\sin x < 1$ so there are only two solutions.

10 a The graphs cross twice between 0° and 360° so there are two solutions.

b $\cos x = \tan x \Rightarrow \cos x = \frac{\sin x}{\cos x}$
Multiply by $\cos x$: $\cos^2 x = \sin x \rightarrow 1 - \sin^2 x = \sin x$
Rearrange: $\sin^2 x + \sin x - 1 = 0$
Use the quadratic formula.
$$\sin x = \frac{-1 \pm \sqrt{1+4}}{2} = 0.618 \text{ or } -1.618$$
$\sin x = -1.618$ has no solution.
The smallest solution of $\sin x = 0.618$ is 38.2°.

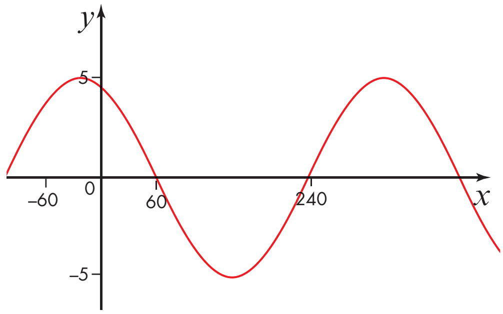
Exam-style questions 6 p 177

1 a $x = 30°$ or $180 - 30 = 150°$

b $2y - 20 = 30 \Rightarrow y = 25°$
$2y - 20 = 150 \Rightarrow y = 85°$

2 Two angles of the parallelogram are 70°, the other two are 110°.
$AC^2 = 10^2 + 15^2 - 2 \times 10 \times 15 \times \cos 110° \Rightarrow AC = 20.7$ cm

3 a



b $\cos(x + 30°) = -0.8$
$x + 30° = 143.1°$
or $360 - 143.1 = 216.9°$
$x = 113.1°$ or $186.9°$

4 The shortest side is opposite the smallest angle, which is 50°.
If the shortest side is x:
$$\frac{x}{\sin 50°} = \frac{35}{\sin 68°} \Rightarrow x = \frac{35\sin 50°}{\sin 68°} = 28.9 \text{ cm}$$

5 Divide by $\cos\theta$: $\tan\theta = 4$
$\Rightarrow \theta = 76.0°$ or $76.0 - 180 = -104.0°$

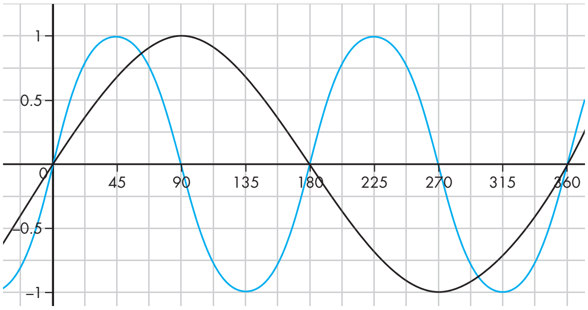
6 $\frac{\sin X}{24.9} = \frac{\sin 50°}{19.5} \Rightarrow \sin X = \frac{24.9\sin 50°}{19.5} = 0.9782$
$X = 78°$ or $180 - 78 = 102°$. There are two possible values for the angle. Carla and Larry have each found one.

7 $\frac{\sin\theta}{\cos\theta} = 2\sin\theta \Rightarrow \sin\theta = 2\sin\theta\cos\theta$
Rearrange: $2\sin\theta\cos\theta - \sin\theta = 0$
$\sin\theta(2\cos\theta - 1) = 0$
Either $\sin\theta = 0$ so $\theta = 0°, 180°$ or $360°$
or $\cos\theta = \frac{1}{2}$ so $\theta = 60°$ or $300°$
The possible solutions are 0°, 60°, 180°, 300° and 360°.

8 Use the cosine rule to find the largest angle; call it A.
$$\cos A = \frac{15^2 + 18^2 - 21^2}{2 \times 15 \times 18} = 0.2$$
$A = 78.46°$
Area of triangle $= \frac{1}{2} \times 15 \times 18 \times \sin 78.46° = 132$ cm²

9 $6(1 - \cos^2 x) = 5 + \cos x$
$6 - 6\cos^2 x = 5 + \cos x$
$\Rightarrow 6\cos^2 x + \cos x - 1 = 0$
Factorise.
$(3\cos x - 1)(2\cos x + 1) = 0$
Either $3\cos x - 1 = 0 \Rightarrow \cos x = \frac{1}{3} \Rightarrow x = 70.5°$ or $289.5°$
or $2\cos x + 1 = 0 \Rightarrow \cos x = -\frac{1}{2} \Rightarrow x = 120°$ or $240°$
There are four solutions.

10 Here are the sketch graphs of $y = \sin x°$ and $y = \sin 2x°$



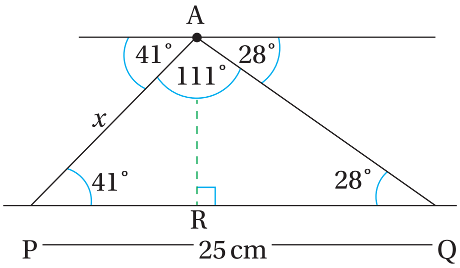
Two obvious solutions, where they cross, are $x = 0$ and $x = 180$

They also cross at $x = 60$ because $\sin 60°$ and $\sin(2 \times 60°) = \sin 120°$ are the same.

By symmetry, they also cross at $x = 360 - 60 = 300$

The four solutions are $x = 0$, 60, 180 or 300

11 You can find the angles of triangle APQ.



Using the sine rule, $\frac{x}{\sin 28°} = \frac{25}{\sin 111°}$ so $x = \frac{25 \sin 28°}{\sin 111°} = 12.57$

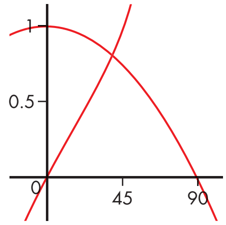
From triangle PAR, width AR = $x\sin 41° = 8.25$ so the width is 8.25 m

12 $\sin^2 x + \cos^2 x = 1$, therefore $\sin^2 x = 1 - \left(-\frac{\sqrt{3}}{4}\right)^2$

$= 1 - \frac{3}{16} = \frac{13}{16}$ and so $\sin x = \pm\frac{\sqrt{13}}{4}$

Then $\tan x = \frac{\sin x}{\cos x} = \pm\frac{\sqrt{13}}{4} \div -\frac{\sqrt{3}}{4} = \pm\frac{\sqrt{13}}{\sqrt{3}}$. This could be written as $\pm\frac{\sqrt{39}}{3}$

13 a



The graphs of $y = \cos x$ and $y = \tan x$ cross between 0° and 90°

b $\tan x = \cos x$ therefore $\frac{\sin x}{\cos x} = \cos x$ which gives $\sin x = \cos^2 x$ and so $\sin x = 1 - \sin^2 x$

Rearranging gives, $\sin^2 x + \sin x - 1 = 0$ Use the quadratic formula: $\sin x = \frac{-1 \pm \sqrt{1+4}}{2} = \frac{-1 \pm \sqrt{5}}{2}$

When $\sin x = \frac{-1+\sqrt{5}}{2}$, this gives $x = 38.2°$ and $x = 180° - 38.2° = 141.8°$

When $\sin x = \frac{-1-\sqrt{5}}{2}$, there are no solutions.

14 a When $x = 0$, $0.35 + 0.2\cos(100x + 60°) = 0.35 + 0.1 = 0.45$

After a short time, say, 0.1 s, the height is $0.35 + 0.2\cos 70° = 0.42$ m, less than 0.45 m.

b When $0.35 + 0.2\cos(100x + 60°) = 0.5$ then $0.2\cos(100x + 60°) = 0.15$ and so $\cos(100x + 60°) = 0.75$ Therefore $100x + 60 = 41.4$ or $360 - 41.4$ or $360 + 41.4$ …

Therefore $x = -0.19$ or 2.59 or 3.41 … The negative value is not appropriate. The formula shows that the height is greater than 0.5 if x is between 2.59 and 3.41 (For example, if $x = 3$ the height is 0.55 m) The time in each revolution is $3.41 - 2.59 = 0.82$ or about 0.8 s.

7 Exponentials and logarithms

Prior knowledge page 180

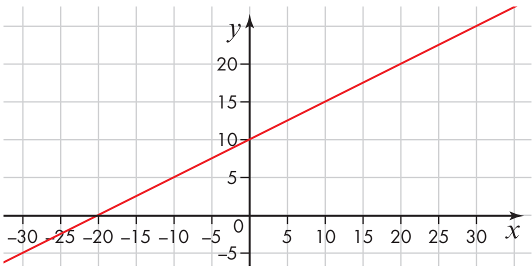
1 **a** 2^4 **b** 2^{-5} **c** $2^{\frac{1}{2}}$

d $\left(2^3\right)^{\frac{1}{2}} = 2^{\frac{3}{2}}$ **e** 2^0

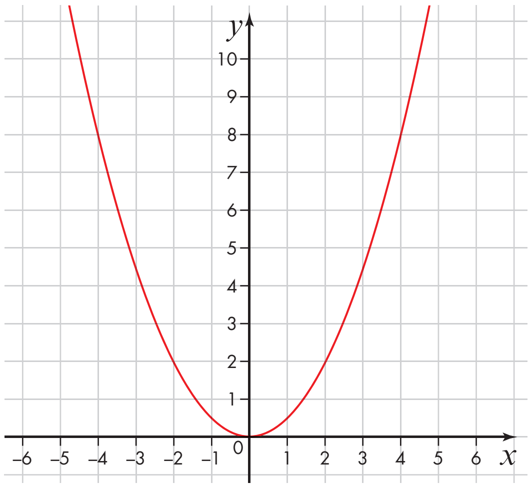
2 **a** n^5 **b** n^{-1}

c n^6 **d** $n^{\frac{1}{2}} \times n^{\frac{1}{3}} = n^{\frac{5}{6}}$

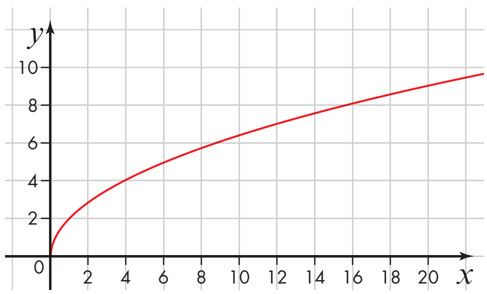
3 **a**



b



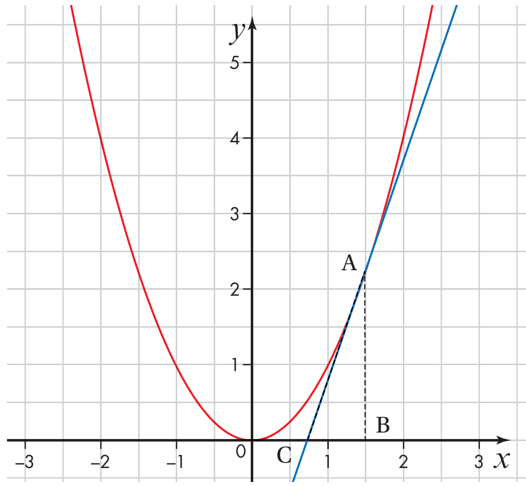
c



4 Gradient $= \frac{20-18}{0-5} = -0.4$

Equation is $y = -0.4x + 20$

5



The straight line is the tangent at (1.5, 2.25).

Using triangle ABC, the gradient is $\frac{AB}{CB} = \frac{2.25}{0.75} = 3$.

A different triangle could be used.

Some variation in the answer is permissible because it is based on a drawing.

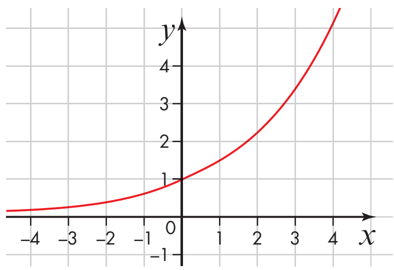
6 New price = 24 000 × 1.04 × 1.04 = £25 958.40

Exercise 7.1A page 183

1 **a**

x	–3	–2	–1	0	1	2	3
$y = 1.5^x$	0.30	0.44	0.67	1	1.5	2.25	3.38

b

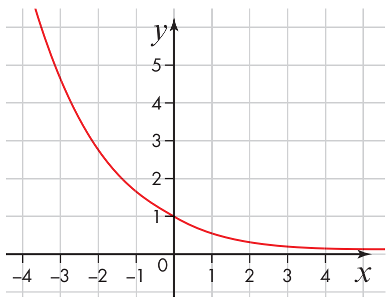


c Where y-coordinate is 3, $x \approx 2.7$.

2 **a**

x	–3	–2	–1	0	1	2	3
$y = 0.6^x$	4.63	2.78	1.67	1	0.6	0.36	0.22

b



c Where y-coordinate is 0.5, $x \approx 1.4$.

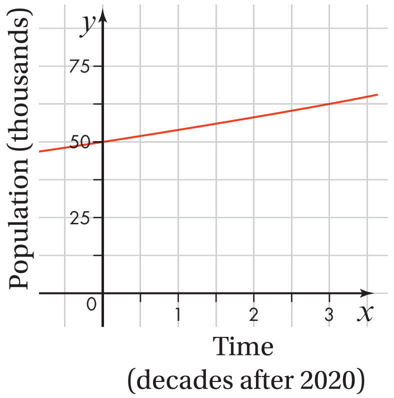
d Where y-coordinate is 3, $x \approx -2.7$.

3 a Multiply by 1.08 repeatedly.

Year	2020	2030	2040	2050
Predicted population	50 000	54 000	58 320	62 986

b $y = 50 \times 1.08^x$

c



d In 2110, $x = 9$

Formula gives $y = 50 \times 1.08^9 = 99.95$

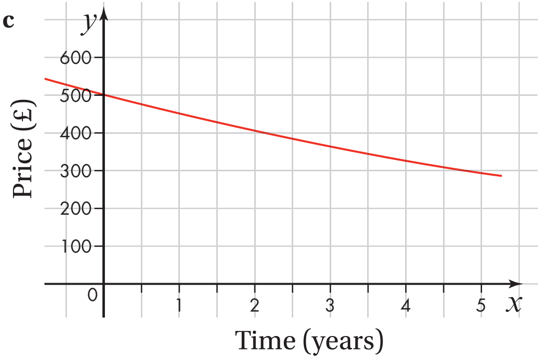
This is double 50.

e Birth rate, death rate and net immigration and emigration rates can all change over time.

4 a $£500 \times 0.9 \times 0.9 = £405$

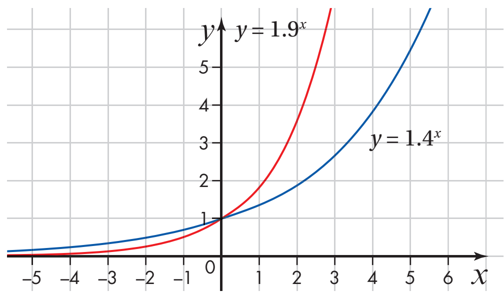
b Multiply the original price by 0.9 for each year.

c



d No. After 5 years, price $= 500 \times 0.9^5 = 295$, which is more than half of 500.

5 a



b When $x > 0$ **c** When $x < 0$

6 For the red graph, when $x = 1$, $y = 3.5$.

This means that the equation is $y = 3.5^x$ $a = 3.5$.

For the green graph, when $x = 1$, $y = 0.5$.

This means that the equation is $y = 0.5^x$ $a = 0.5$.

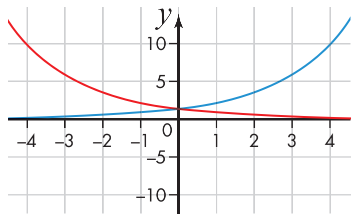
7 When $x = 0$, $y = K$. From the graph, $K = 500$.

From the graph, when $x = 1$, $y = 650$

$500a = 650$

$a = 650 \div 500 = 1.3$

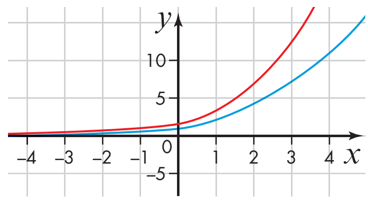
8 a



b A reflection in the y-axis.

c If n is a positive integer, a reflection in the y-axis maps the graph of $y = n^x$ onto the graph of $y = \left(\frac{1}{n}\right)^x$

9 a



b Because x is replaced by $x + 1$, it is a translation of $\begin{bmatrix} -1 \\ 0 \end{bmatrix}$.

c $y = 2^{x+1} = 2 \times 2^x$

A stretch, scale factor 2, from the x-axis will map the graph of $y = 2^x$ onto the graph of $y = 2^{x+1}$.

Exercise 7.2A page 186

1 a $\log 10^3 = 3$ **b** $\log 10^6 = 6$

c $\log 10^{-3} = -3$ **d** $\log 10^{-6} = -6$

2 a $\log 10^{\frac{1}{2}} = \frac{1}{2}$ **b** $\log 10 = 1$

c $\log 10^{\frac{3}{2}} = \frac{3}{2}$ **d** $\log 10^{-\frac{1}{2}} = -\frac{1}{2}$

3 a $\log(2 \times 10) = \log 2 + \log 10 = 0.3010 + 1 = 1.3010$

b $\log(2 \div 10) = \log 2 - \log 10 = 0.3010 - 1 = -0.6990$

c $\log(10 \div 2) = \log 10 - \log 2 = 1 - 0.3010 = 0.6990$

d $\log 2^{-\frac{1}{2}} = -\frac{1}{2} \times \log 2 = -0.1505$

4 **a** $\log(8 \times 10) = \log 8 + \log 10 = 0.9031 + 1 = 1.9031$

b $\log(12 \div 8) = \log 12 - \log 8$
$= 1.0792 - 0.9031 = 0.1761$

c $\log(12 \times 8 \div 10) = \log 12 + \log 8 - \log 10$
$= 1.0792 + 0.9031 - 1 = 0.9823$

d $\log \frac{8}{12} = \log(8 \div 12) = \log 8 - \log 12$
$= 0.9031 - 1.0792 = -0.1761$

5 **a** $2k$

b $\log 100 + \log x = 2 + k$

c $\log 1 - \log x = 0 - k = -k$

d $\log 10 - \log x^{\frac{1}{2}} = 1 - \frac{1}{2}k$

6 **a** $\log x + \log y^2 = k + 2h$

b $\log x^{\frac{1}{2}} - \log y = \frac{1}{2}k - h$

c $\log 100 + \log x^2 - \log y^3 = 2 + 2k - 3h$

d $\log 1 - \log x - \log y = -k - h$

7 **a** $x = 10^2 = 100$

b $2y = 10^2$
$y = 50$

c $\log z = 10^2 = 100$
$z = 10^{100}$

8 **a** $\log 4 = \log 2^2 = 2 \log 2$

b $\log 5 = \log \frac{10}{2} = \log 10 - \log 2 = 1 - \log 2$

c $\log 6 = \log(2 \times 3) = \log 2 + \log 3$

d log 7 cannot be found in this way.

e $\log 8 = \log 2^3 = 3 \log 2$

f $\log 9 = \log 3^2 = 2 \log 3$

g log 11 cannot be found in this way.

h $\log 12 = \log(2^2 \times 3) = 2 \log 2 + \log 3$

Exercise 7.2B page 188

1 **a** $\log_2 2^6 = 6$ **b** $\log_5 5^3 = 3$

c $\log_3 3^5 = 5$ **d** $\log_4 4^4 = 4$

2 **a** $\log_9 9^{\frac{1}{2}} = \frac{1}{2}$

b $27 = 9 \times 3 = 9 \times 9^{\frac{1}{2}}$

$\log_9 9^{\frac{3}{2}} = \frac{3}{2}$

c $\log_9 9^{-2} = -2$

d $3 = 9^{\frac{1}{2}}$

$\sqrt{3} = \sqrt{9^{\frac{1}{2}}} = 9^{\frac{1}{4}}$

$\log_9 9^{\frac{1}{4}} = \frac{1}{4}$

3 $2 \log_a 4 = \log_a 16$

$\log_a 8 - \log_a 2 = \log_a 4$

$\log_a \frac{1}{2} - \log_a \frac{1}{6} = \log_a 3$

$3 \log_a 2 = \log_a 8$

4 **a** 7 **b** $\log_2 \frac{1}{32} = -\log_2 32 = -5$

c $\frac{1}{2} \log_2 512 = 4.5$ **d** $\frac{1}{3} \log_2 256 = \frac{8}{3}$

5 **a** $\log_3 (3 \times 5) = \log_3 3 + \log_3 5 = 1 + c$

b $\log_3 \frac{1}{5} = \log_3 1 - \log_3 5 = -c$

c $\log_3 5^3 = 3c$

d $5 = 3^c = \left(9^{\frac{1}{2}}\right)^c = 9^{\frac{1}{2}c}$

$\log_9 5 = \frac{1}{2}c$

6 **a** $n^3 = 216$ so $n = 6$ **b** $n^{\frac{1}{2}} = 4$ so $n = 4^2 = 16$

c $n^0 = 1$ so n can be any positive integer.

d $n^{1.5} = 27$

$n \times \sqrt{n} = 27$

$n = 9$

7 $a = n^{3.5}$

$b = n^{4.5} = n^{1+3.5} = n \times n^{3.5} = na$

8 **a** **i** $b = a^x$ **ii** $a = b^y$

b $a = b^y$

Substitute for b: $a = (a^x)^y = a^{xy}$

This means that $xy = 1$ and the result follows.

Exercise 7.3A page 190

1 **a** $x \log 3 = \log 11$ so

$x = \frac{\log 11}{\log 3} = 2.183$

b $x \log 4 = \log 175$ so

$x = \frac{\log 175}{\log 4} = 3.726$

c $x \log 12 = \log 6$.

$x = \frac{\log 6}{\log 12} = 0.721$

2 **a** 1.322

b 3.887

c 5.674

3 **a** $1.8^t = \frac{750}{200} = 3.75$ so

$t = \frac{\log 3.75}{\log 1.8} = 2.249$

b $0.87^t = \frac{4500}{7000} = 0.6429$ so

$t = \frac{\log 0.6429}{\log 0.87} = 3.173$

c $1.04^t = \frac{123}{95} = 1.2947$ so

$t = \frac{\log 1.2947}{\log 1.04} = 6.586$

4 a If it takes t years, $1.09^t = 1.5$

$t = \frac{\log 1.5}{\log 1.09} = 4.70$

4.7 years

b If it takes t years, $1.09^t = 2$

$t = \frac{\log 2}{\log 1.09} = 8.04$

Just over 8 years

5 a $(x + 2) \log 4 = \log 90$

$x = \frac{\log 90}{\log 4} - 2 = 1.246$

b $(2x + 1) \log 6 = \log 35$

$2x + 1 = \frac{\log 35}{\log 6}$

$x = 0.4921$

c $(4x - 3) \log 15 = \log 8$

$4x - 3 = \frac{\log 8}{\log 15}$

$x = 0.942$

6 a $y = 10 \times 1.5^t$ **b** $10 \times 1.5^3 = 33.75\text{ m}^2$

c When $10 \times 1.5^t = 80$:

$1.5^t = 8$

$t \log 1.5 = \log 8$

$t = \frac{\log 8}{\log 1.5} = 5.1$

5.1 weeks

d The multiplier is now 1.2.

When $10 \times 1.2^t = 80$:

$1.2^t = 8$

$t \log 1.5 = \log 8$

$t = \frac{\log 8}{\log 1.2} = 11.4$

11.4 weeks

e Changes in temperature and light conditions can affect the growth rate.

Growth will not continue when the pond is fully covered.

7 a $1.65 \times 1.05 = 1.7325$ million

$1.7325 \times 1.05 = 1.82$ million (to 3 s.f.)

b 1.65×1.05^t

c When $1.65 \times 1.05^t = 2.5$:

$1.05^t = \frac{2.5}{1.65} = 1.5151$

$t = \frac{\log 1.5151}{\log 1.05} = 8.52$

8.5 years

d The number cannot increase beyond the capacity of the airport.

Changes in travel patterns could make the model incorrect.

8 a If the annual multiplier is m, then $m^2 = 2$.

$m = \sqrt{2} = 1.4142$

If $1.4142^t = 100$, then $t = \frac{\log 100}{\log 1.4142} = 13.3$

Just over 13 years

b This time, $m^{1.5} = 2$.

$m = 2^{\frac{2}{3}} = 1.5874$

If $1.5874^t = 100$, then $t = \frac{\log 100}{\log 1.5874} = 9.97$

10 years

c The size of an atom puts a limit on how small components can be made.

Exercise 7.4A page 194

1 a $\log y = \log 250x^2$

$\log y = \log 250 + 2 \log x$

b 2

c $(0, \log 250)$

2 $\log V = \log \frac{4}{3}\pi r^3 = \log \frac{4}{3}\pi + \log r^3$

$= \log \frac{4}{3}\pi + 3 \log r$

The graph is a straight line with a gradient of 3.

3 a $\log F = \log \frac{Gm_1m_2}{r^2} = \log Gm_1m_2 - \log r^2$

$= \log Gm_1m_2 - 2 \log r$

So the graph is a straight line.

b Gradient = –2

c Intercept is $(0, \log Gm_1m_2)$.

4 a $\log P = \log Ac^t = \log A + t \log c$

This is a straight line graph with a gradient of $\log c$.

b $\log c = 0.0128$ so $c = 10^{0.0128} = 1.03$

$\log A = 1.97$

$A = 10^{1.97} = 93.3$

c 1.03 is the multiplier for an annual percentage increase of 3%.

d Changes in birth rates, life expectancy and immigration rates or emigration rates will change the annual rate of growth.

5 **a** Gradient $= \frac{6.5-1.5}{10} = 0.5$

b $\log v = \log ar^n = \log a + n\log r$. This is the equation of a straight line and the intercept on the y-axis is $\log a$. From the graph, $\log a = 1.5$.

c The gradient is n so $n = 0.5$

$\log a = 1.5$ so $a = 10^{1.5} = 31.6$

d $v = 31.6 \times 100^{0.5} = 31.6 \times 10 = 316$

6 **a** $\log x^2y = \log k$

$2\log x + \log y = \log k$

$\log y = -2\log x + \log k$

A graph of $\log y$ against $\log x$ is a straight line with a gradient of 2.

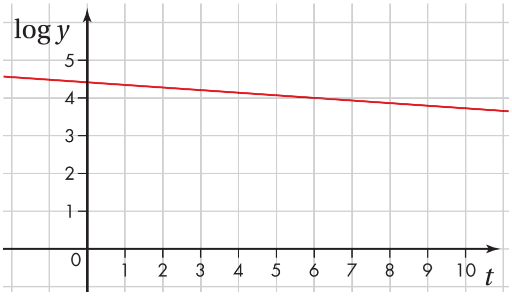
b $\log k = 4.6$

$k = 10^{4.6} = 40\,000$ to 2 s.f.

7 $\log y = \log A + t\log c$

A graph of $\log y$ against t is a straight line.

Age (t years)	2	4	6	8	10
Value (£y)	18 490	13 675	10 114	7480	5533
log y	4.267	4.136	4.005	3.874	3.743



c From the graph, $\log A = 4.4$

$A = 10^{4.4} = 25\,000$ to 2 s.f.

The gradient of the graph is $-0.066 = \log c$

$c = 10^{-0.066} = 0.86$ to 2 s.f.

(Note: values from a graph could vary slightly.)

8 The gradient is $-\frac{2}{3}$ and the intercept on the y-axis is 2. If $\log y = 2$, then $y = 100$.

The equation is $\log y = -\frac{2}{3}\log x + 2$

$\log y = -\frac{2}{3}\log x + \log 100 = \log 100 - \log x^{\frac{2}{3}}$

$y = \frac{100}{x^{\frac{2}{3}}}$ or $x^{\frac{2}{3}}y = 100$

Exercise 7.5A — page 200

1 **a** 7.39 **b** 1.65 **c** 0.368

d 6.57 **e** 0.0498

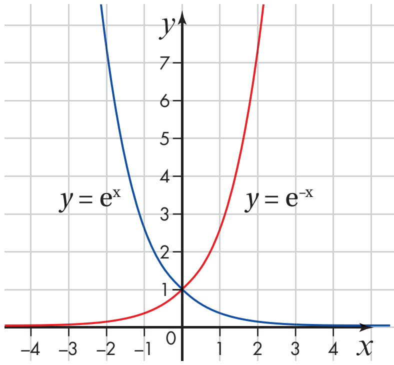
2 **a** To 5 decimal places the values are:

i 2.704 81 **ii** 2.718 15 **iii** 2.718 28

b The answers increase and approach the value of e as n increases.

c The value of the expression approaches e.

3 **a**



b Reflection in the y-axis

4 **a** $f'(x) = 4e^x$ **b** $f'(x) = -e^{-x}$

c $f'(x) = 0.5e^{0.5x}$ **d** $f(x) = 30e^{3x}$

5 Gradient $= e^x$

a $e^0 = 1$ **b** e^2 **c** 2

6 Gradient $= 3e^{3x}$

a $3e^0 = 3$ **b** $3e^6$ **c** $3 \times 2 = 6$

7 **a** One method is to find where each curve crosses the y-axis.

$y = e^x + 1$ is C.

$y = e^{x+1}$ is A.

$y = (e+1)^x$ is B.

b $y = e^x + 1$ is C and a translation of $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$y = e^{x+1}$ is A and a translation of $\begin{bmatrix} -1 \\ 0 \end{bmatrix}$

c A is $y = e^{x+1} = e \times e^x$ so it is a stretch of $y = e^x$ by a scale factor of e.

8 **a** If $x = 0$, $y = e^{x+2} = e^2$. The intercept is $(0, e^2)$.

b A translation of $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$ will map $y = e^{x+4}$ onto $y = e^{(x-2)+4}$, which is $y = e^{x+2}$

9 **a** 2.718 055…

b e = 2.718 2818… so the value of the sum is a bit less than e.

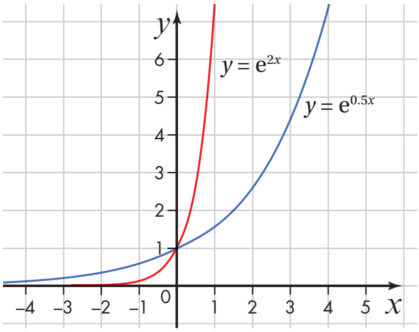
c $\frac{1}{7!} = \frac{1}{5040}$

Adding this gives 2.718 253… which is closer to e.

d It suggests that adding more terms will get even closer to e.

Adding $\frac{1}{8!} = \frac{1}{40320}$ gives 2.718 278… which is closer.

10 a Sketches should look like this:



b $y = e^{0.5x} = e^{2\left(\frac{x}{4}\right)}$

Replacing x by $\frac{x}{4}$ in the equation $y = e^{2x}$ results in a stretch of scale factor 4 from the y-axis which maps it onto $y = e^{0.5x}$.

11 Rearrange: $2x^2 + ex - e^2 = 0$

This is a quadratic in x.

Factorise: $(2x - e)(x + e) = 0$

Either $2x - e = 0$ or $x + e = 0$

Solutions are $x = \frac{e}{2}$ or $x = -e$

Exercise 7.6A page 202

1 a $\ln a + \ln b = 7.5$ **b** $4 \times \ln a = 12$

c $\ln b - 2 \times \ln a = -1.5$ **d** $\frac{1}{2}(\ln a + \ln b) = 3.75$

2 a $x = \ln 2000 = 7.60$

b $-x = \ln 0.03$

$x = 3.51$

c $2x - 5 = \ln 125$

$x = \frac{5 + \ln 125}{2} = 4.91$

d $-\frac{1}{2}x^2 = \ln 0.5$

$x = \pm\sqrt{-2 \ln 0.5} = \pm 1.18$

3 a $0.5x = e^4$

$x = 109$

b $4x + 2 = e^{3.5}$

$x = \frac{e^{3.5} - 2}{4} = 7.78$

c $x = \frac{1}{2}e^{-4} = 0.009\,16$

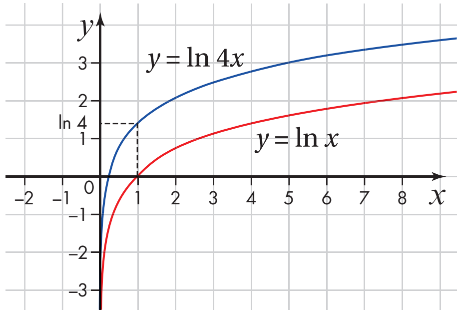
d $x = \frac{120}{e^5} = 0.809$

4 $\ln \frac{1}{a} = \ln 1 - \ln a = -\ln a$ because $\ln 1 = 0$.

5 a $y = \ln 4x = \ln 4 + \ln x$

The graph is a translation of $y = \ln x$ by $\begin{bmatrix} 0 \\ \ln 4 \end{bmatrix}$.

b



6 a Either $\ln 20 + t = \ln 100$ or $t = \ln \frac{100}{20} = \ln 5$

$t = 1.61$

b $-t = \ln \frac{35}{40}$

$t = 0.134$

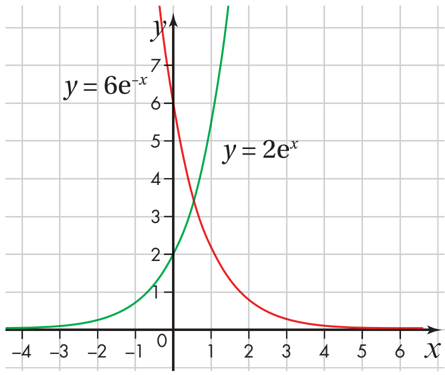
c $3t = \ln \frac{8000}{250}$

$t = 1.16$

d $-0.85t = \ln \frac{14.8}{32.5}$

$t = \frac{\ln 0.4554}{-0.85} = 0.925$

7 a



b Where they cross, $2e^x = 6e^{-x}$

$e^{2x} = 3$

$2x = \ln 3$

$x = \frac{\ln 3}{2} = 0.5493$;

$y = 2e^{0.5493} = 3.464$

Coordinates, to 3 s.f., are (0.549, 3.46).

8 a B is (ln 2, 2)

b The gradient is the y-coordinate = 2

c The gradient is $\frac{1}{2}$

d The gradient of $y = \ln x$ at $(x, \ln x)$ is $\frac{1}{x}$ or you could say $\frac{dy}{dx} = \frac{1}{x}$.

9 **a** $y^2 + y = 6$ and this is a quadratic equation in y.

b Rearrange and factorise: $y^2 + y - 6 = 0$

$(y + 3)(y - 2) = 0$

$y = -3$ or 2

c Either $e^x = -3$ or $e^x = 2$

$e^x = -3$ has no solution because e^x is always positive.

If $e^x = 2$, then $x = \ln 2 = 0.693$ to 3 s.f.

10 Write $y = e^x$:

$2y^2 - 9y + 4 = 0$

$(2y - 1)(y - 4) = 0$

$y = \frac{1}{2}$ or 4

If $e^x = \frac{1}{2}$, then $x = \ln \frac{1}{2} = -0.693$ to 3 s.f.

If $e^x = 4$, then $x = \ln 4 = 1.39$ to 3 s.f.

11 **a** $\ln(e + e^2) = \ln(e(1 + e)) = \ln e + \ln(1 + e)$

$= 1 + \ln(1 + e)$

b $\ln(e^2 - e^4) = \ln(e^2(1 - e^2) = \ln e^2(1 + e)(1 - e)$

$= \ln e^2 + \ln(1 + e) + \ln(1 - e)$

$= 2 + \ln(1 + e) + \ln(1 - e)$

Exercise 7.7A page 206

1 **a** When $t = 0$, $v = 5000$

The initial value is £5000.

b **i** When $t = 1$, $v = 5000e^{0.18} = 5986$

Value is £5986

ii When $t = 5$, $v = 5000e^{0.9} = 12\,298$

Value is £12 298

iii When $t = 10$, $v = 5000e^{1.8} = 30\,248.24$

Value is £30 248

2 **a** Where the graph meets the y-axis, value is £2000.

b From graph, when $y = 4$, $x \approx 3.15$

3.1 or 3.2 years

c $\frac{2.5 - 2}{2} \times 100 = 25\%$

d $\frac{3.75 - 3}{3} \times 100 = 25\%$

e This will also be 25%, within the accuracy of reading the graph.

f They are the same. The annual rate of increase is constant.

3 **a** Current value is N. When $t = 5$, $x = Ne^{-0.75} = 0.472N$

This is slightly less than $0.5N$.

b If the model is valid, it will fall by just over 50% again, to less than 25% of the current value.

4 **a**

Number of years from now	0	10	20	30
Population	50	61.07	74.59	91.11

b $\frac{11.07}{50} \times 100 = 22.1\%$

$\frac{13.59}{61.07} \times 100 = 22.1\%$

$\frac{16.52}{74.59} \times 100 = 22.1\%$

In each case the percentage increase is 22.1%.

5 Where the graph meets the y-axis, $A = 40$

When $t = 20$, $y = 15$

$15 = 40e^{-20k}$

Divide by 40: $e^{-20k} = 0.375$

$-20k = \ln 0.375$

$k = \frac{\ln 0.375}{-20} = 0.049$

6 **a**

Days	0	2	5	10	15
Mass remaining (mg)	5.0	4.4	3.7	2.7	2.0

b

c From graph, when $m = 2.5$, $t = 11.5$ or 11.6 days

d $2.5 = 5e^{-0.06t}$

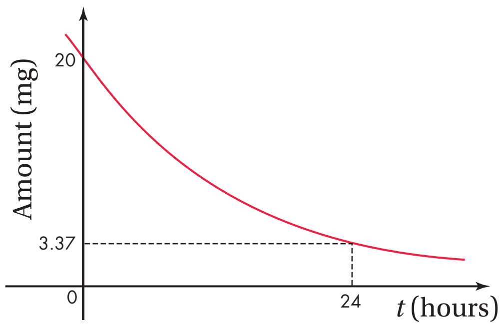
$e^{-0.06t} = 0.5$

$t = \frac{\ln 0.5}{-0.06} = 11.55$ days

e $2 \times 11.55 = 23.10$ days

7 **a** When $t = 24$, amount $= 20e^{-1.68} = 3.73$ mg

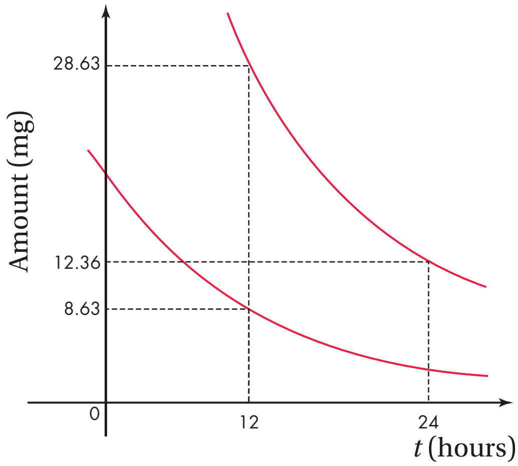
b



c 12 hours after the second dose was administered, the amount of it left in the bloodstream is $20e^{-0.07 \times 12} = 8.63$ mg.

The total amount of the drug is $8.63 + 3.73 = 12.36$ mg

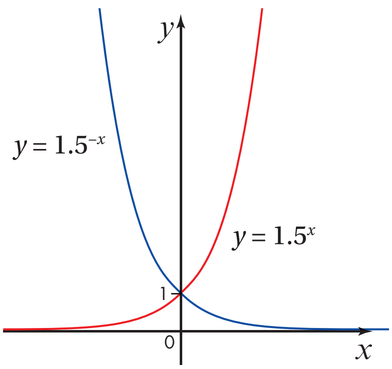
d



Practice questions 7 page 209

1 **a** If $y = 4$, $x \approx 3.4$

b



2 $3t + 8 = 5^{2.1}$

$$t = \frac{5^{2.1} - 8}{3} = 7.12$$

3 **a** If $x = 0$, $y = e^3$

Crosses y-axis at $(0, e^3)$.

b If $x = -2$, $k = e^{0.5 \times (-2) + 3} = e^2$ or 7.389

c $100 = e^{0.5h + 3}$

$0.5h + 3 = \ln 100$

$$h = \frac{\ln 100 - 3}{0.5} = 3.21$$

4 **a** When $x = 0$, $y = c$

From the graph, $c = 2.5$

b The gradient is $-0.5y = -0.5 \times 2.5 = -1.25$

5 **a** $\log_9 3x = \log_9 3 + \log x = \log_9 \sqrt{9} + k = \frac{1}{2} + k$

b $x = 9^k = (3^2)^k = 3^{2k}$ so $\log_3 x = \log_3 3^{2k} = 2k$

6 $\ln 10r = 0.5t + 1$

$$t = \frac{\ln 10r - 1}{0.5} \text{ or } t = 2(\ln 10r - 1)$$

7 **a** £10 000

b Value after 1 year $= 10e^{0.3}$

Percentage growth in first year $= \frac{10e^{0.3} - 10}{10} \times 100$

$= 35.0\%$

The annual growth is 35.0% for any year.

c The value after one month is $10e^{\frac{0.3}{12}} = 10.253$.

The monthly percentage increase is

$$\frac{10.253 - 10}{10} \times 100 = 2.53\%$$

8 **a** When $t = 0$, $y = 375$

$375 = Ae^0 = A$

b When $t = 4$, $y = 945$

$945 = 375e^{4k}$

$$e^{400k} = \frac{945}{375} = 2.52$$

$$k = \frac{\ln 2.52}{4} = 0.002\,31$$

c In 1600, $t = 2$ and Sam's value for the population is $y = 375e^{2k}$

$= 375e^{0.462} = 595$

This is not very close to the actual value of 540.

9 **a** $\log y = \log k + t \log a$

A graph of $\log y$ against t crosses the $\log y$ axis at $\log k$ so $\log k = 3.30$.

$k = 10^{3.3} = 2000$ to 3 s.f.

b The gradient is $\log a$.

$$\log a = \frac{4.65 - 3.30}{3} = 0.45$$

$a = 10^{0.45} = 2.82$ to 3 s.f.

c The population may not have enough food or space to keep growing at the same rate.

10 $590 = 20 \times 15^n$

$15^n = \frac{590}{20} = 29.5$

$n \log 15 = \log 29.5$

$n = \frac{\log 29.5}{\log 15} = 1.25$ to 3 s.f.

(You could use ln instead of log.)

11 a Multiply both sides by e^x: $e^{2x} = 4$

$2x = \ln 4$

$x = \frac{\ln 4}{2} = 0.693$ to 3 s.f.

b Multiply both sides by e^x:

$e^{2x} - 3e^x - 4 = 0$

This is a quadratic in e^x.

Factorise.

$(e^x - 4)(e^x + 1) = 0$

$e^x = 4$ or $e^x = -1$

The second of these has no solution.

If $e^x = 4$, then $x = \ln 4$ or 1.386 to 3 d.p.

12 Let $y = \log_a b$ This can be written as $a^y = b$ Take the yth root: $a = b^{\frac{1}{y}}$ Therefore $\log_b a = \frac{1}{y}$ and so $\log_a b = \frac{1}{\log_b a}$

13 $\log_{10}(8\sqrt{5}) = \log_{10} 8 + \log_{10}\sqrt{5} = \log_{10}(2^3) + \frac{1}{2}\log_{10} 5$

$= 3\log_{10} 2 + \frac{1}{2}\log_{10}\frac{10}{2} = 3\log_{10} 2 + \frac{1}{2}(\log_{10} 10 - \log_{10} 2)$

$= \frac{1}{2}(1 + 5\log_{10} 2)$

14 a When $t = 0$, $y = A = 200$; when $t = 5$, $yA = 200e^{5k}$ $= 148$ so $e^{5k} = \frac{148}{200} = 0.74$

$5k = \ln 0.74$ and $k = \frac{\ln 0.74}{5} = -0.060$

b $y = 200e^{-0.060t}$ When $t = 10$, $y = 200e^{-0.60} = 110$ and this is less than the recorded value.

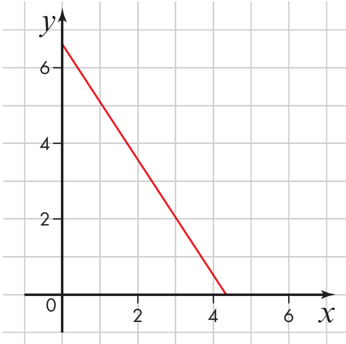
c $Ae^{5k} = 148$ and $Ae^{10k} = 121$; Divide the two: $\frac{Ae^{10k}}{Ae^{5k}} = \frac{121}{148}$ so $e^{5k} = 0.818$

$5k = \ln 0.818$ and $k = -0.040$ Hence $y = Ae^{-0.040t}$ and when $t = 5$, $148 = Ae^{-0.20}$ and therefore $A = 181$; the revised formula is $y = 181e^{-0.040t}$

d When t is large, the temperature approaches 0. This will only be the case if the temperature of the surroundings is 0 °C.

15 $\log y = \log a + n \log x$ so draw a graph of $\log y$ against $\log x$

$\log_{10} x$	2	2.9	3.7
$\log_{10} y$	3.6	2.25	1.05



From the graph, the gradient is –1.5 so $n = -1.5$

The intercept on the y-axis is 6.6 so $\log a = 6.6$ and $a = 10^{6.6} = 4\,000\,000$

16 $2^{2x} + 8 = 8 \times 2^x$ Write $y = 2^x$ and then $y^2 - 8y + 8 = 0$

Solve with the quadratic formula: $= \frac{8 \pm \sqrt{64 - 4 \times 8}}{2}$; $= \frac{8 + \sqrt{32}}{2}$; $y = 6.828$ or 1.172; either $2^x = 6.828$ or $2^x = 1.172$

Take logs: either $x = \frac{\log 6.828}{\log 2} = 2.772$ or $x = \frac{\log 1.172}{\log 2} = 0.228$

8 Differentiation

Prior knowledge — page 213

1 **a** 7 **b** −0.3
c $\frac{1}{2}$ **d** −2

2 **a** $2x^2 - 8x$ **b** $x^3 + 3x^2$
c $x^2 + 4x - 5x - 20 = x^2 - x - 20$
d $9x^2 + 3x + 3x + 1 = 9x^2 + 6x + 1$

3 **a** x^{-3} **b** $x^{\frac{1}{2}}$ **c** $x^{-\frac{1}{2}}$
d $x^{\frac{1}{3}}$ **e** $x^{\frac{3}{2}}$

Exercise 8.1A — page 214

1 **a** Speed: 250 m min^{-1}
b Acceleration: −2.5 m s^{-2}
c Rate of refuelling: 25 l min^{-1}
d Rate of increase in volume: 0.7 l min^{-1}
e Rate of loss of charge: 4 percentage points per hour

2 **a** 5 m s^{-2} **b** From the graph: about 10 m s^{-2}
c From the graph: about 3.3 m s^{-2}

3 **a** **i** 5 m **ii** 20 m **iii** 45 m
b 20 m s^{-1}
c Estimates from the graph:
i 10 m s^{-1} **ii** 35 m s^{-1}

4 **a** From the gradients: 20 m s^{-1} upwards and 5 m s^{-1} downwards
b 5 m s^{-1} upwards and 20 m s^{-1} downwards
c From a tangent at that point: about 10 m s^{-1}

5 From tangents: gradients are approximately
a 1.3 m s^{-1} **b** 0.5 m s^{-1}

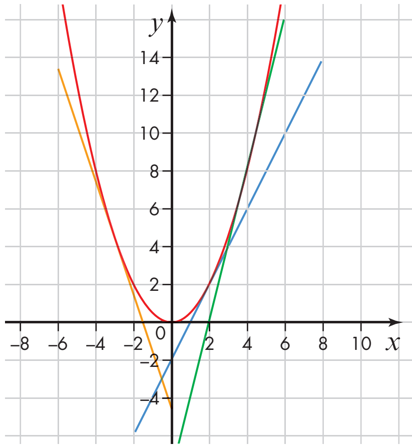
Exercise 8.2A — page 219

1 $\frac{dy}{dx} = 2x$
a $y = 16$ and $\frac{dy}{dx} = 8$ **b** $y = 6.25$ and $\frac{dy}{dx} = 5$
c $y = 1.44$ and $\frac{dy}{dx} = -2.4$ **d** $y = 1225$ and $\frac{dy}{dx} = 70$

2 **a** 3 **b** −1
c 26 **d** −7.6

3 $x^2 = 49$ so $x = 7$ or −7
The gradients are 14 and −14.

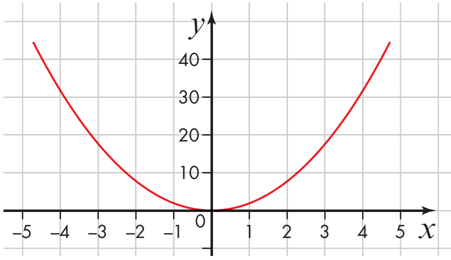
4 **a, b**



c From graph: about 2, −3 and 4
d $\frac{dy}{dx} = x$
e Tangents at other points should confirm this.

5 In both cases $k = 0.5 \times 100 = 50$.
You saw in **question 4** that the gradient is the same as the x-coordinate. At (10, 50) the gradient is 10 and at (−10, 50) the gradient is −10.

6 **a**



Notice that the scales on the axes are different.

b If you draw some tangents you should be able to confirm that $\frac{dy}{dx} = 4x$.

Exercise 8.2B — page 221

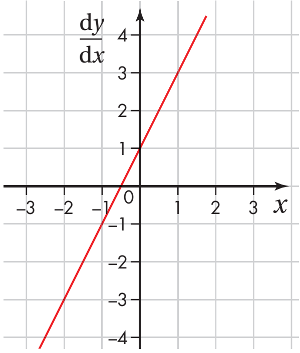
1 **a** 3 **b** $6x$ **c** $6x + 4$

2 **a** $2x + 4$ **b** $2x - 4$ **c** $4 - 2x$

3 **a** $t = x^2 + 6x$, $\frac{dy}{dx} = 2x + 6$
b $y = x^2 - 2x - 3$, $\frac{dy}{dx} = 2x - 2$
c $y = 4x^2 + 12x + 9$, $\frac{dy}{dx} = 8x + 12$

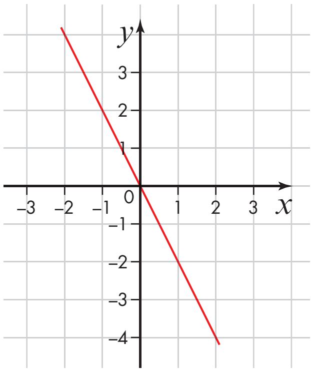
4 **a** $\frac{dy}{dx} = 2x + 1$
b **i** At (0, −6) **ii** When $x = 0$, $\frac{dy}{dx} = 1$
c At (2, 0) and (−3, 0), the gradients are 5 and −5.

d



5 **a** $f'(x) = -2x$ **b** $f'(-1) = 2$ and $f'(3) = -6$

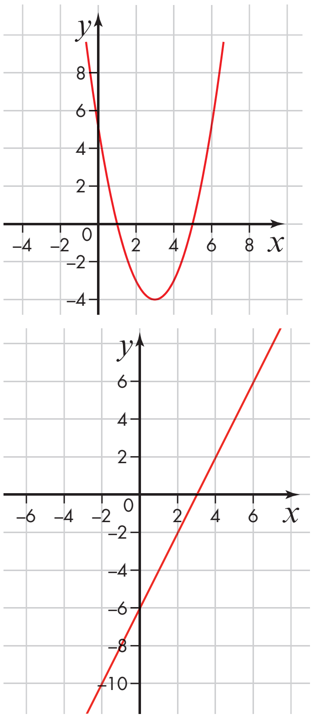
c



6 **a** $y = x^2 - 6x + 5$ so $f'(x) = 2x - 6$

b $f'(1) = -4$ and $f'(11) = 16$

c



7 $\frac{dy}{dx} = 30x - 30$

a When $x = 1$, then $\frac{dy}{dx} = 0$

b When $x = 5$, then $\frac{dy}{dx} = 120$

8 $\frac{dy}{dx} = 10x$

When $x = 1$, then $\frac{dy}{dx} = 10$

When $x = 3.5$, then $\frac{dy}{dx} = 35$

9 $\frac{dy}{dx} = 30 - 10x$

When $x = 2$, then $\frac{dy}{dx} = 10$

10 $y = kx(x - 4) = kx^2 - 4kx$

$\frac{dy}{dx} = 2kx - 4k$

When $x = 4$, $\frac{dy}{dx} = 2$

$2k \times 4 - 4k = 2$

$8k - 4k = 2$

$4k = 2$

$k = \frac{1}{2}$ or 0.5

Exercise 8.3A page 224

1 **a** $\frac{dy}{dx} = 2x$ so the gradient at P is $2 \times 3 = 6$

b Gradient of PQ $= \frac{9.61 - 9}{3.1 - 3} = 6.1$

c $\frac{3.05^2 - 9}{3.05 - 3} = 6.05$

d $\frac{3.01^2 - 9}{3.01 - 3} = 6.01$

e As Q gets closer to P, the gradient of the chord gets closer to the gradient of the tangent.

2 $\frac{dy}{dx} = 2x$ so the gradient at P is $2 \times 2 = 4$

The gradient of PQ $= \frac{4 - 3.61}{2 - 1.9} = 3.9$, which is a little smaller than 4.

3 **a** $f(x + \delta x) - f(x) = 2(x + \delta x)^2 - 2x^2$

$= 2x^2 + 4x\delta x + 2(\delta x)^2 - 2x^2$

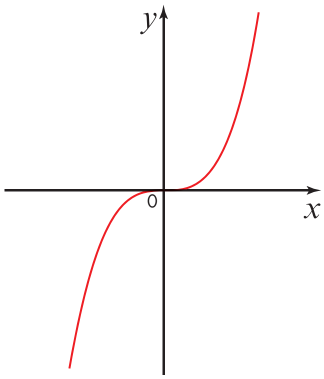
$= 4x\delta x + 2(\delta x)^2$

b The gradient of the chord is

$$\frac{f(x + \delta x) - f(x)}{\delta x} = \frac{4x\,\delta x + 2(\delta x)^2}{\delta x} = 4x + 2\delta x$$

c $f'(x) = \lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x} = \lim_{\delta x \to 0} (4x + 2\delta x) = 4x$

4 **a**



b $f(x + \delta x) - f(x) = 3x^2\delta x + 3x(\delta x)^2 + (\delta x)^3$

c $\frac{f(x + \delta x) - f(x)}{\delta x} = \frac{3x^2\delta x + 3x(\delta x)^2 + (\delta x)^3}{\delta x}$

$= 3x^2 + 3x\delta x + (\delta x)^2$

d As $\delta x \to 0$, $3x\delta x + (\delta x)^2 \to 0$ and so $\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = 3x^2$

e $\frac{dy}{dx} = 3x^2$

f x^2 is always positive unless $x = 0$, when $x^2 = 0$.

The gradient is never negative.

Exercise 8.4A — page 226

1 **a** $2 \times 3x^2 = 6x^2$ **b** $0.5 \times 4x^3 = 2x^3$

c $0.1 \times 5x^4 = 0.5x^4$ **d** $50 \times 3x^2 = 150x^2$

2 **a** $3x^2 + 8x - 8$ **b** $6x^2 - 10x + 6$

3 **a** $4x^3 + 16x$ **b** $5x^4 - 30x^2 + 2$

4 **a** $3x^2 - 4x$

b **i** When $x = 2$, $3x^2 - 4x = 12 - 8 = 4$

ii When $x = -1$, $3x^2 - 4x = 3 + 4 = 7$

c Visual check

5 **a** $0.5 \times 4x^3 - 2 \times 2x + 1 = 2x^3 - 4x + 1$

b **i** When $x = -1$, $2x^3 - 4x + 1 = -2 + 4 + 1 = 3$

ii When $x = 1$, $2x^3 - 4x + 1 = 2 - 4 + 1 = -1$

iii When $x = 2$, $2x^3 - 4x + 1 = 16 - 8 + 1 = 9$

c Visual check

6 **a** $y = 2x^3 + 5x^2$

$\frac{dy}{dx} = 6x^2 + 10x$

b $y = x(x^2 - 8x + 16) = x^3 - 8x^2 + 16x$

$\frac{dy}{dx} = 3x^2 - 16x + 16$

c $y = x^3 + x^2 + x + 1$

$\frac{dy}{dx} = 3x^2 + 2x + 1$

7 **a** $f(x) = x^3 + 3x^2$

$f'(x) = 3x^2 + 6x$

b **i** $f'(-3) = 27 - 18 = 9$

ii $f'(-2) = 12 - 12 = 0$

iii $f'(1) = 3 + 6 = 9$

8 $y = x(x^2 - 6x + 8) = x^3 - 6x^2 + 8x$

$\frac{dy}{dx} = 3x^2 - 12x + 8$

When $x = 0$, $\frac{dy}{dx} = 8$

When $x = 2$, $\frac{dy}{dx} = 12 - 24 + 8 = -4$

When $x = 4$, $\frac{dy}{dx} = 48 - 48 + 8 = 8$

9 **a** $\frac{dy}{dx} = 0.012x^2 - 0.6x + 7.5$

When $x = 10$, $\frac{dy}{dx} = 1.2 - 6 + 7.5 = 2.7$, which is the correct value.

b When $x = 20$, $\frac{dy}{dx} = 4.8 - 12 + 7.5 = 0.3$

The speed is 0.3 m s^{-1}.

Exercise 8.5A — page 229

1 **a** $y = x^{-2}$

$\frac{dy}{dx} = -2x^{-3} = -\frac{2}{x^3}$

b $\frac{dy}{dx} = 2 \times (-3)x^{-4} = -\frac{6}{x^4}$

c $\frac{dy}{dx} = \frac{1}{3}x^{-\frac{2}{3}}$ **d** $\frac{dy}{dx}\, 4 \times \frac{5}{2}x^{\frac{5}{2}-1} = 10x^{\frac{3}{2}}$

2 **a** $10 \times \frac{1}{2}x^{\frac{1}{2}-1} = 5x^{-\frac{1}{2}}$ or $\frac{5}{\sqrt{x}}$

b $50 \times (-1)\, x^{-2} = -\frac{50}{x^2}$

c $20x - 10 \times (-2)\, x^{-3} = 20x + \frac{20}{x^3}$

3 **a** $f(x) = 24x^{-1}$

$f'(x) = -24x^{-2} = -\frac{24}{x^2}$

b **i** $f'(6) = -\frac{24}{6^2} = -\frac{2}{3}$ **ii** $f'(4) = -\frac{24}{4^2} = -\frac{3}{2}$

iii $f'(-2) = -\frac{24}{(-2)^2} = -6$ **iv** $f'(24) = -\frac{24}{24^2} = -\frac{1}{24}$

c If $x \neq 0$, then x^2 is positive and the gradient $-\frac{24}{x^2}$ is negative.

4 **a** $y = x^{\frac{1}{2}}$

$\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$

i When $x = 4$, $\frac{dy}{dx} = \frac{1}{2\sqrt{4}} = \frac{1}{4}$

ii When $x = 9$, $\frac{dy}{dx} = \frac{1}{2\sqrt{9}} = \frac{1}{6}$

iii When $x = 100$, $\frac{dy}{dx} = \frac{1}{2\sqrt{100}} = \frac{1}{20}$

b $\frac{1}{2\sqrt{x}} = \frac{1}{2}$

$x = 1$ and the coordinates are $(1, 1)$.

c $\frac{1}{2\sqrt{x}} = 1$

$\sqrt{x} = \frac{1}{2}$

$x = \frac{1}{4}$ and the coordinates are $(\frac{1}{4}, \frac{1}{2})$.

5 $y = 4x^{\frac{1}{3}}$

$\frac{dy}{dx} = \frac{4}{3}x^{-\frac{2}{3}}$

When $x = 1$, $\frac{dy}{dx} = \frac{4}{3} \times 1^{-\frac{2}{3}} = \frac{4}{3}$

When $x = 8$, $\frac{dy}{dx} = \frac{4}{3} \times 8^{-\frac{2}{3}} = \frac{4}{3} \times \frac{1}{4} = \frac{1}{3}$

6 **a** $y = \frac{1}{2}x + 2x^{-1}$

$\frac{dy}{dx} = \frac{1}{2} - 2x^{-2} = \frac{1}{2} - \frac{2}{x^2}$

When = 2, $\frac{dy}{dx} = \frac{1}{2} - \frac{2}{2^2} = \frac{1}{2} - \frac{1}{2} = 0$

When = –2, $\frac{dy}{dx} = \frac{1}{2} - \frac{2}{(-2)^2} = \frac{1}{2} - \frac{1}{2} = 0$

b When $x = 0.5$, $\frac{dy}{dx} = \frac{1}{2} - \frac{2}{0.5^2} = \frac{1}{2} - 8 = -7\frac{1}{2}$

When x = 4, $\frac{dy}{dx} = \frac{1}{2} - \frac{2}{4^2} = \frac{1}{2} - \frac{1}{8} = \frac{3}{8}$

c $\frac{dy}{dx} = \frac{1}{2} - \frac{2}{x^2}$

If x is large, $\frac{2}{x^2}$ is a small positive number and the gradient is close to $\frac{1}{2}$.

The larger x is, the closer the gradient is to $\frac{1}{2}$.

7 **a** $f(x) = 2 + \frac{5}{x} = 2 + 5x^{-1}$

$f'(x) = -5x^{-2} = -\frac{5}{x^2}$

b **i** $f'(2) = -\frac{5}{2^2} = -\frac{5}{4}$

ii $f'(10) = -\frac{5}{10^2} = -\frac{1}{20} = -0.05$

c $f'(x) = -\frac{5}{x^2} = -5$

$x^2 = 1$

$x = 1$ or -1

Points are (1, 7) and (–1, –3).

Exercise 8.5B page 232

1 **a** $\frac{dV}{dr} = \frac{4}{3}\pi \times 3r^2 = 4\pi r^2$

b When $r = 3$, $\frac{dV}{dr} = 4\pi \times 3^2 = 36\pi$

2 **a** $\frac{dh}{dt} = 20 - 10t$

b and **c**

i When $t = 0$, $\frac{dh}{dt} = 20$ m s^{-1}.

The stone is moving upwards.

ii When $t = 2$, $\frac{dh}{dt} = 20 - 20 = 0$ m s^{-1}.

This means the stone is at its highest point.

iii When $t = 3$, $\frac{dh}{dt} = 20 - 20 = -10$ m s^{-1}.

The minus sign means the stone is returning to the ground.

3 **a** $p = 6q^{\frac{1}{2}}$

$\frac{dp}{dq} = 3q^{-\frac{1}{2}} = \frac{3}{\sqrt{q}}$

When $q = 4$, $\frac{dp}{dq} = \frac{3}{\sqrt{4}} = \frac{3}{2}$

b When $p = 4$, $4 = 6\sqrt{q}$ so $\sqrt{q} = \frac{4}{6} = \frac{2}{3}$

$\frac{dp}{dq} = \frac{3}{\frac{2}{3}} = \frac{9}{2}$

4 **a** Volume = $\pi r^2 h$

$\pi r^2 h = 1000$

$h = \frac{1000}{\pi r^2}$

b $h = \frac{1000}{\pi} r^{-2}$

$\frac{dh}{dr} = \frac{1000}{\pi} \times (-2) r^{-3} = -\frac{2000}{\pi r^3}$

c When $r = 6$, $\frac{dh}{dr} = -\frac{1}{2} - \frac{2000}{\pi r^3} = -\frac{2000}{\pi \times 6^3} = 2.950$ to 3 s.f.

Exercise 8.6A page 236

1 **a** $\frac{dy}{dx} = 2x - 12$

At a stationary point, $\frac{dy}{dx} = 0$.

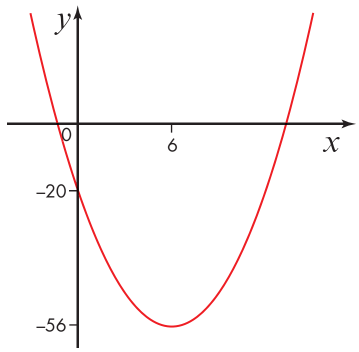
$2x - 12 = 0$

$x = 6$ so $y = 6^2 - 12 \times 6 - 20 = -56$

Coordinates are (6, –56).

b $\frac{d^2y}{dx^2} = 2$ which is positive when $x = 6$ (in fact it is always positive) so the point is a minimum point.

c



2 **a** $\frac{dy}{dx} = 6 - 4x$

At a stationary point, $\frac{dy}{dx} = 0$.

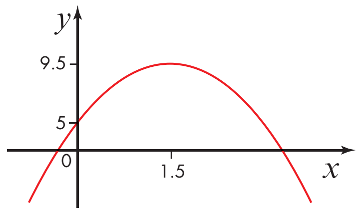
$6 - 4x = 0$

$x = 1\frac{1}{2}$ so $y = 5 + 6 \times 1\frac{1}{2} - 2 \times \left(1\frac{1}{2}\right)^2 = 9.5$

Coordinates are (1.5, 9.5).

b $\frac{d^2y}{dx^2} = -4$ which is negative when $x = 1.5$ (in fact it is always negative) so the point is a maximum point.

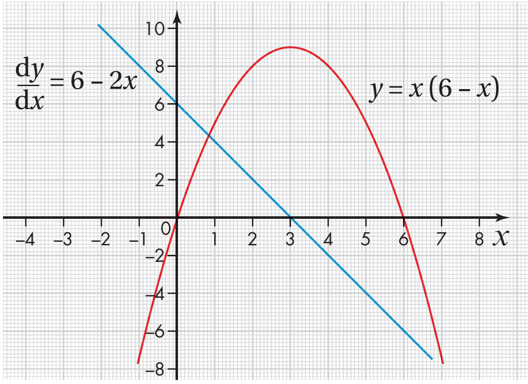
c



3 **a, b**

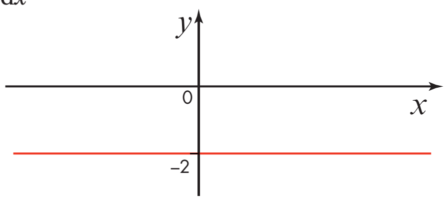
$y = 6x - x^2$

$\frac{dy}{dx} = 6 - 2x$



c Where the graph of $\frac{dy}{dx}$ crosses the x-axis gives the x-coordinate of a turning point.

d $\frac{d^2y}{dx^2} = -2$, a constant value



4 **a** $f'(x) = 6x^2 - 18x + 12$

At a stationary point, $f'(x) = 0$.

$6x^2 - 18x + 12 = 0$

Divide by 6 and factorise: $x^2 - 3x + 2 = 0$

$(x - 1)(x - 2) = 0$

$x = 1$ or 2

$f(1) = 2 - 9 + 12 + 8 = 13$

and $f(2) = 16 - 36 + 24 + 8 = 12$

The stationary points are (1, 13) and (2, 12).

b $f''(x) = 12x - 18$

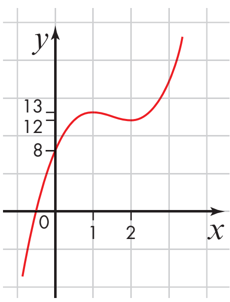
$f''(1) = 12 - 18 = -6$

(1, 13) is a maximum point because $f''(x)$ is negative at $x = 1$.

$f''(2) = 24 - 18 = 6$

(2, 12) is a minimum point because $f''(x)$ is positive at $x = 2$.

c



5 **a** $\frac{dy}{dx} = 3x^2 - 12x - 180$

At a stationary point, $\frac{dy}{dx} = 0$.

$3x^2 - 12x - 180 = 0$

Divide by 3 and factorise.

$x^2 - 4x - 60 = 0$

$(x - 10)(x + 6) = 0$

$x = 10$ or -6

When $x = 10$, $y = 10^3 - 6 \times 10^2 - 180 \times 10 = -1400$

$x = -6$, $y = (-6)^3 - 6 \times (-6)^2 - 180 \times (-6) = 648$

The stationary points are (10, –1400) and (– 6, 648).

b $\frac{d^2y}{dx^2} = 6x - 12$

When $x = 10$, $\frac{d^2y}{dx^2} = 48$ so (10, –1400) is a minimum point because $\frac{d^2y}{dx^2}$ is positive for $x = 10$.

When $x = -6$, $\frac{d^2y}{dx^2} = -48$ so (– 6, 648) is a maximum point because $\frac{d^2y}{dx^2}$ is negative for $x = -6$.

6 $\frac{dy}{dx} = 4x^3 - 4x$

At a stationary point, $\frac{dy}{dx} = 0$.

$4x^3 - 4x = 0$

$x(x - 1)(x + 1) = 0$

$x = 0$, 1 or -1

The stationary points are (0, 0), (1, –1) and (–1, –1).

$\frac{d^2y}{dx^2} = 12x^2 - 4$

When $x = 0$, $\frac{d^2y}{dx^2} = -4$ and is negative so (0, 0) is a maximum point.

When $x = 1$, $\frac{d^2y}{dx^2} = 8$ and is positive so (1, –1) is a minimum point.

When $x = -1$, $\frac{d^2y}{dx^2} = 8$ and is positive so (–1, –1) is a minimum point.

7 $y = 10x^{-1} + \frac{1}{4}x$

$\frac{dy}{dx} = -10x^{-2} + \frac{1}{4}$

At a stationary point, $\frac{dy}{dx} = 0$.

$-10x^{-2} + \frac{1}{4} = 0$

Rearrange.

$10x^{-2} = \frac{1}{4}$

$40 = x^2$

$x = \sqrt{40}$

$y = \frac{10}{\sqrt{40}} + \frac{\sqrt{40}}{4} = \frac{10\sqrt{40}}{40} + \frac{\sqrt{40}}{4}$

$= \frac{\sqrt{40}}{4} + \frac{\sqrt{40}}{4} = \frac{\sqrt{40}}{2}$

$= \frac{2\sqrt{10}}{2} = \sqrt{10}$

The exact coordinates are $(\sqrt{40}, \sqrt{10})$.

8 $\frac{dy}{dx} = 6 - x$

At a maximum point, $\frac{dy}{dx} = 0$.

$6 - x = 0$

$x = 6$

$\frac{d^2y}{dx^2} = -1$ which is negative for any value of x so this is a maximum point.

If $x = 6$, $y = 10 + 6 \times 6 - 0.5 \times 6^2 = 10 + 36 - 18 = 28$

The maximum speed is 28 m s^{-1}.

9 a $\frac{dy}{dx} = 23 - 10x$

When the height is a maximum, $\frac{dy}{dx} = 0$.

$23 - 10x = 0$

$x = 2.3$

$\frac{d^2y}{dx^2} = -10 < 0$ so this is a maximum point.

When $x = 2.3$, $y = 23 \times 2.3 - 5 \times 2.3^2 = 26.45$

The maximum height is 26.45 m.

b Air resistance will reduce the maximum height of the ball.

10 a Volume $= x^2h = 1000$

Rearrange: $h = \frac{1000}{x^2}$

b The total surface area is

$2x^2 + 4xh = 2x^2 + 4x \times \frac{1000}{x} = 2x^2 + \frac{4000}{x^2}$

c If $a = 2x^2 + \frac{4000}{x} = 2x^2 + 4000x^{-1}$ then

$\frac{da}{dx} = 4x - 4000x^{-2} = 4x - \frac{4000}{x^2}$

When the surface area has a minimum value, $\frac{da}{dx} = 0$.

$4x = \frac{4000}{x^2}$

$x^3 = 1000$

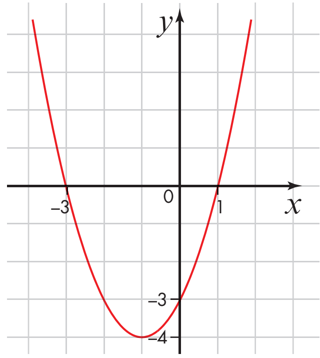
$x = 10$

$\frac{d^2a}{dx^2} = 4 + 8000x^{-3}$

When $x = 10$, $\frac{d^2a}{d^2x} = 4 + \frac{8000}{10^3} = 12$ which is positive, so the surface area has a minimum value. In fact, in this case the cuboid is a cube.

Exercise 8.7A — page 239

1 a This parabola crosses the x-axis at 1 and –3.



b $y = x^2 + 2x - 3$

$\frac{dy}{dx} = 2x + 2$

c When $x = 3$, $\frac{dy}{dx} = 2 \times 3 + 2 = 8$

The equation of the tangent is $y - 12 = 8(x - 3)$

$y - 12 = 8x - 24$

$y = 8x - 12$

d The gradient of the normal is $-\frac{1}{8}$

The equation of the normal at T is

$y - 12 = -\frac{1}{8}(x - 3)$

$8y - 96 = -x + 3$

$x + 8y = 99$

e When $x = -3$, $\frac{dy}{dx} = 2 \times -3 + 2 = -4$, so the tangent is

$y - 0 = -4(x + 3)$

$y + 4x = -12$

The gradient of the normal is $\frac{1}{4}$ and the equation is $y = \frac{1}{4}(x + 3)$ or $4y = x + 3$.

2 a When $x = -1$, $y = (-1)^3 - 4 \times (-1)^2 = -5$ so $(-1, -5)$ is on the curve.

b $\frac{dy}{dx} = 3x^2 - 8x$

When $x = -1$, $\frac{dy}{dx} = 3 \times (-1)^2 - 8 \times (-1) = 3 + 8 = 11$

c The equation of the tangent is $y + 5 = 11(x + 1)$

$y + 5 = 11x + 11$

$y = 11x + 6$

d The equation of the normal is $y + 5 = -\frac{1}{11}(x + 1)$

$11y + 55 = -x - 1$

$11y + x = -56$

3 $y = 12x^{-2}$

$\frac{dy}{dx} = -24x^{-3}$

When $x = 1$, $\frac{dy}{dx} = -24$

The gradient of the normal is $\frac{1}{24}$

The equation of the normal is

$y - 12 = \frac{1}{24}(x - 1)$

$24y - 288 = x - 1$

$24y = x + 287$

4 a $y = 12x^{\frac{1}{2}}$

$\frac{dy}{dx} = 6x^{-\frac{1}{2}} = \frac{6}{\sqrt{x}}$

When $x = 4$, $\frac{dy}{dx} = \frac{6}{\sqrt{4}} = 3$

Gradient of normal $= -\frac{1}{3}$

Equation of normal is

$y - 24 = -\frac{1}{3}(x - 4)$

$3y - 72 = -x + 4$

$x + 3y = 76$

b When $x = 100$, $\frac{dy}{dx} = \frac{6}{\sqrt{100}} = \frac{3}{5}$

Gradient of tangent $= \frac{3}{5}$

Equation of tangent is

$y - 120 = \frac{3}{5}(x - 100)$

$5y - 600 = 3x - 300$

$5y = 3x + 300$

5 a $\frac{dy}{dx} = -2x$

When $x = 2$, $\frac{dy}{dx} = -4$ and the equation of the tangent is $y - 6 = -4(x - 2)$

$y - 6 = -4x + 8$

$y + 4x = 14$

b Draw a diagram. Only the tangent is required.

The tangent crosses the x-axis at 3.5 and the area of the triangle is $\frac{1}{2} \times 14 \times 3.5 = 24.5$

6 a $\frac{dy}{dx} = \frac{2}{3}x^{-\frac{1}{3}}$

When $x = 8$, $\frac{dy}{dx} = \frac{2}{3} \times 8^{-\frac{1}{3}} = \frac{2}{3} \times \frac{1}{2} = \frac{1}{3}$

The equation of the tangent is $y - 4 = \frac{1}{3}(x - 8)$

$3y - 12 = x - 8$

$3y = x + 4$

b You need to find the coordinates of the point.

When $\frac{dy}{dx} = \frac{1}{6}$, $\frac{2}{3}x^{-\frac{1}{3}} = \frac{1}{6}$

$4x^{-\frac{1}{3}} = 1$

$4 = x^{\frac{1}{3}}$

$x = 4^3 = 64$

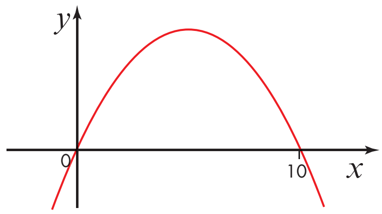
$y = 64^{\frac{2}{3}} = 4^2 = 16$

Equation of tangent is $y - 16 = \frac{1}{6}(x - 64)$

$6y - 96 = x - 64$

$6y = x + 32$

7 a $y = x(10 - x)$ so it crosses the x-axis at 0 and 10.



b $\frac{dy}{dx} = 10 - 2x$

When $x = 3$, $\frac{dy}{dx} = 10 - 2 \times 3 = 4$

Equation of tangent is $y - 21 = 4(x - 3)$

$y - 21 = 4x - 12$

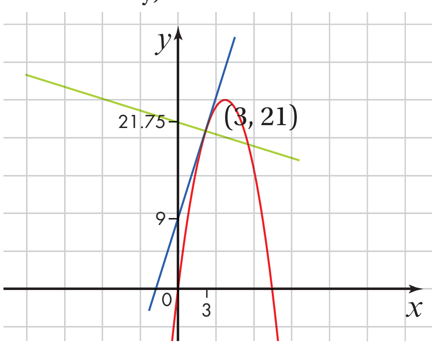
$y = 4x + 9$

Equation of normal is $y - 21 = -\frac{1}{4}(x - 3)$

$4y - 84 = -x + 3$

$4y + x = 87$

Sketch the tangent and the normal (the curve is not necessary).



The tangent meets the y-axis at 9. The normal meets the y-axis at $\frac{87}{4} = 21\frac{3}{4}$

The area of the triangle is $\frac{1}{2}\left(21\frac{3}{4} - 9\right) \times 3 = 19\frac{1}{8}$ or 19.125

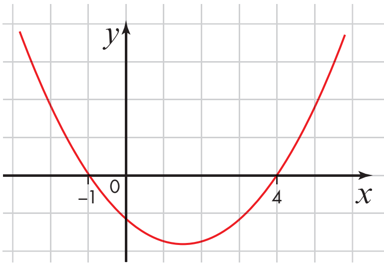
Exam-style questions 8 — page 240

1 **a** $y = 2x + 8x^{-1}$

$\frac{dy}{dx} = 2 - 8x^{-2} = 2 - \frac{8}{x^2}$

b When $x = 4$, $\frac{dy}{dx} = 2 - \frac{8}{4^2} = 1.5$

2 The gradient is 0 when $x = -1$ or 4.



3 **a** You cannot find the derivative of a product by finding the derivative of each term separately and multiplying the results.

b $y = 3x^2 - 3x + 2x - 2 = 3x^2 - x - 2$

$\frac{dy}{dx} = 6x - 1$

4 **a** $y = x^{\frac{1}{2}}(x + 4) = x^{\frac{3}{2}} + 4x^{\frac{1}{2}}$

$\frac{dy}{dx} = \frac{3}{2}x^{\frac{1}{2}} + 2x^{-\frac{1}{2}}$

When $x = 9$, $\frac{dy}{dx} = \frac{3}{2} \times 9^{\frac{1}{2}} + 2 \times 9^{-\frac{1}{2}}$

$= \frac{9}{2} + \frac{2}{3} = \frac{31}{6} = 5\frac{1}{6}$

5 **a** $y = x^4 - 8x^2$

$\frac{dy}{dx} = 4x^3 - 16x$

b At a stationary point, $4x^3 - 16x = 0$.

Divide by 4 and factorise.

$x(x^2 - 4) = 0$

$x(x - 2)(x + 2) = 0$

$x = 0, 2 \text{ or } -2$

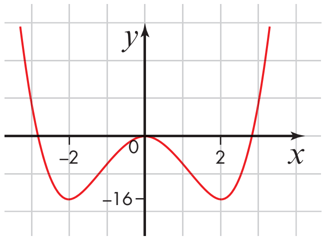
When $x = 0$, $y = 0$

When $x = 2$, $y = 2^4 - 8 \times 2^2 = -16$

When $x = -2$, $y = (-2)^4 - 8 \times (-2)^2 = -16$

Stationary points are (0, 0), (2, –16) and (–2, –16).

c



6 $\frac{dy}{dx} = 6x^2 - 12x - 12$

If the gradient is 6 then $6x^2 - 12x - 12 = 6$

$x^2 - 2x - 3 = 0$

$(x - 3)(x + 1) = 0$

$x = 3$ or -1

When $x = 3$, $y = 2 \times 27 - 6 \times 9 - 12 \times 3 + 4 = -32$

When $x = -1$, $y = -2 - 6 + 12 + 4 = 8$

The points are (3, –32) and (–1, 8).

7 **a** $\frac{dy}{dx} = 0.4x - 0.03x^2$

When $x = 10$, $\frac{dy}{dx} = 4 - 3 = 1$

Speed is 1 m s^{-1} upwards.

b At the highest point, $\frac{dy}{dx} = 0$.

$0.4x - 0.03x^2 = 0$

$x(0.4 - 0.03x) = 0$

Either $x = 0$, which is when the drone starts, or $0.4 - 0.03x = 0$

$x = \frac{0.4}{0.03} = 13\frac{1}{3}$

This is $\frac{2}{3}$ of 20 seconds, showing the ascent takes twice as long as the descent.

8 **a** $y = x^2 + 250x^{-1}$

$\frac{dy}{dx} = 2x - 250x^{-2}$

b At a stationary point, $2x - 250x^{-2} = 0$.

$2x^3 = 250$

$x^3 = 125$

$x = 5$

$\frac{d^2y}{dx^2} = 2 + 500x^{-3}$

When $x = 5$, $\frac{d^2y}{dx^2} = 2 + 500 \times 5^{-3} = 6 > 0$ so this is a minimum point.

When $x = 5$, $y = 5^2 + \frac{250}{5} = 75$

The minimum value is 75.

9 $y = x + 10x^{-1}$

$\frac{dy}{dx} = 1 - 10x^{-2}$

At the minimum point, $1 - 10x^{-2} = 0$.

$x^2 = 10$

Take the positive root so $x = \sqrt{10}$

$y = \sqrt{10} + \frac{10}{\sqrt{10}} = \sqrt{10} + \sqrt{10} = 2\sqrt{10}$

The exact coordinates are $(\sqrt{10}, 2\sqrt{10})$.

10 a The other side is of length $240 - 2x$.

The area $A = x(240 - 2x)$.

b $A = 240x - 2x^2$

$\frac{dA}{dx} = 240 - 4x$

For maximum area, $\frac{dA}{dx} = 240 - 4x = 0$.

$x = 60$

$\frac{d^2A}{dx^2} = -4 < 0$ so this will be a maximum.

Maximum area = 60 × 120 = 7200 m²

11 a $f(x + \delta x) - f(x) = 2(x + \delta x)^2 - 2x^2$

$= 2(x^2 + 2x\delta x + (\delta x)^2) - 2x^2$

$= 2x^2 + 4x\delta x + 2(\delta x^2 - 2x)^2 = 4x\delta x + 2(\delta x)^2$

b $\frac{f(x + \delta x) - f(x)}{\delta x} = 4x + 2\delta x$

$f'(x) = \lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x} = \lim_{\delta x \to 0} (4x + 2\delta x) = 4x$

12 $\frac{dy}{dx} = 2x + 4$

When $x = 2$, $\frac{dy}{dx} = 2 \times 2 + 4 = 8$

The equation of the tangent is $y - 7 = 8(x - 2)$

$y = 8x - 9$

13 a Volume of a cone, $V = \frac{1}{3}\pi r^2 h$

$h = 60 - r$ so $V = \frac{1}{3}\pi r^2(60 - r)$

b $V = \frac{1}{3}\pi(60r^2 - r^3)$

$\frac{dV}{dr} = \frac{1}{3}\pi(120r - 3r^2)$

When the volume is a maximum, $\frac{dv}{dr} = 0$.

$\frac{1}{3}\pi(120r - 3r^2) = 0$

$120r - 3r^2 = 0$

Divide by 3 and factorise.

$r(40 - r) = 0$

$r = 0$ or 40

$\frac{d^2V}{dr^2} = \frac{1}{3}\pi(120 - 6r)$

When $r = 40$, $\frac{d^2V}{dr^2} = \frac{1}{3}\pi(120 - 240) < 0$ so this is a maximum point.

c The maximum volume is $\frac{1}{3}\pi 40^2(60 - 40) = \frac{32000\pi}{3}$

14 $y = x^2 - 4x - 5$

$\frac{dy}{dx} = 2x - 4$

When $x = 6$, $\frac{dy}{dx} = 8$

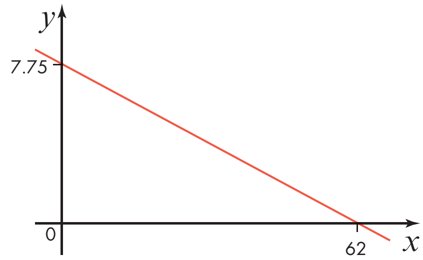
The gradient of the normal is $-\frac{1}{8}$.

The equation of the normal is $y - 7 = -\frac{1}{8}(x - 6)$

$8y - 56 = -x + 6$

$8y + x = 62$

This crosses the x-axis at 62 and the y-axis at $\frac{62}{8} = 7.75$.



Area of triangle = $\frac{1}{2} \times 62 \times 7.75 = 240.25$

15 Assume the point is $\left(a, \frac{20}{a}\right)$ $y = 20x^{-1}$ so

$\frac{dy}{dx} = -20x^{-2} = -\frac{20}{x^2}$

At the point $\frac{dy}{dx} = -\frac{20}{a^2}$ and the equation of the tangent is $y - \frac{20}{a} = -\frac{20}{a^2}(x - a)$

Where the tangent meets the x-axis, $y = 0$ so

$-\frac{20}{a} = -\frac{20}{a^2}(x - a)$; $a = x - a$; $x = 2a$

Where the tangent meets the y-axis, $x = 0$ so

$y - \frac{20}{a} = -\frac{20}{a^2} \times (-a) = \frac{20}{a}$ and $y = \frac{40}{a}$

The area of the triangle is $\frac{1}{2} \times 2a \times \frac{40}{a} = 40$ and it is independent of a.

16 $f(x) = x^3 - 2x^2$ and $f(x + \delta x) = (x + \delta x)^3 - 2(x + \delta x)^2$

$= x^3 + 3x^2\delta x + 3x(\delta x)^2 + (\delta x)^3 - 2x^2 - 4x\delta x - 2(\delta x)^2$

So $\frac{f(x + \delta x) - f(x)}{\delta x}$

$= \frac{3x^2\delta x + 3x(\delta x)^2 + (\delta x)^3 - 4x\delta x - 2(\delta x)^2}{\delta x}$

$= 3x^2 + 3x\delta x + (\delta x)^2 - 4x - 2\delta x$

As $\to 0$, three terms $\to 0$ and

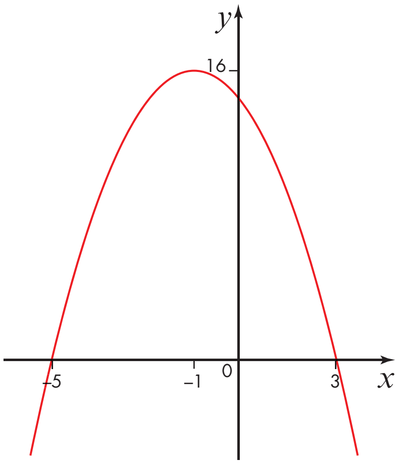
$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x} = 3x^2 - 4x$

9 Integration

Prior knowledge — page 244

1 $4x^{-\frac{1}{2}}$

2 The graph crosses the x-axis at 3 and –5.



3 a $12x^2 - 6$

b $y = \frac{1}{2}x + 2x^{-1}$

$\frac{dy}{dx} = \frac{1}{2} - 2x^{-2}$

c $y = 6x^{\frac{1}{3}}$

$\frac{dy}{dx} = 2x^{-\frac{2}{3}}$

Exercise 9.1A — page 246

1 a $\frac{6}{2}x^2 + c = 3x^2 + c$ b $2x^2 + 2x + c$

c $\frac{7}{2}x^2 - 5x + c$ d $3x - 2x^2 + c$

2 a $\frac{2}{5}x^5 + c$ b $\frac{2}{-3}x^{-3} = -\frac{2}{3}x^{-3} + c$

c $\frac{5}{4}x^4 + c$ d $\int 5x^{-3}\,dx = \frac{5}{-2}x^{-2} + c$

3 a $2x^3 - 2x^2 + c$ b $\frac{2}{5}x^5 - \frac{5}{2}x^2 + 10x + c$

c $4x^2 - \frac{10}{3}x^3 + c$

4 a $\int x^{\frac{1}{2}}\,dx = \frac{2}{3}x^{\frac{3}{2}} + c$ b $2 \times \frac{2}{5}x^{\frac{5}{2}} + c = \frac{4}{5}x^{\frac{5}{2}} + c$

c $3 \times 2x^{\frac{1}{2}} + c = 6x^{\frac{1}{2}} + c$

d $\int 10x^{-\frac{1}{2}}\,dx = 20x^{\frac{1}{2}} + c$ or $20\sqrt{x} + c$

5 a $\int x^2 - 4x\,dx = \frac{1}{3}x^3 - 2x^2 + c$

b $\int x^3 - 8x^2\,dx = \frac{1}{4}x^4 - \frac{8}{3}x^3 + c$

c $\int 10x^{-3}\,dx = \frac{10}{-2}x^{-2} + c = -5x^{-2} + c$ or $-\frac{5}{x^2} + c$

6 a $\int 4x^{\frac{1}{2}}\,dx = 4 \times \frac{2}{3}x^{\frac{3}{2}} + c = \frac{8}{3}x^{\frac{3}{2}} + c$

b $\int \sqrt{4x}\,dx = \int 2\sqrt{x}\,dx = 2 \times \frac{2}{3}x^{\frac{3}{2}} + c = \frac{4}{3}x^{\frac{3}{2}} + c$

c $\int 4x^{-\frac{1}{2}}\,dx = 4 \times 2x^{\frac{1}{2}} + c = 8x^{\frac{1}{2}} + c$ or $8\sqrt{x} + c$

7 $f(x) = \int 2x^3 - 2x\,dx = \frac{2}{4}x^4 - x^2 + c = \frac{1}{2}x^4 - x^2 + c$

8 a You cannot integrate a product by integrating each term separately and multiplying the results.

b First multiply out the brackets.

$\int 2x^2 - x - 1\,dx = \frac{2}{3}x^3 - \frac{1}{2}x^2 - x + c$

9 a $\int 4 - 2x^{-3}\,dx = 4x + x^{-2} + c$

b $\int 2x^{\frac{1}{2}} + 8x^{-\frac{1}{2}}\,dx = 2 \times \frac{2}{3}x^{\frac{3}{2}} + 16x^{\frac{1}{2}} + c$

$= \frac{4}{3}x^{\frac{3}{2}} + 16x^{\frac{1}{2}} + c$

c $\int 4x^{-2} + 2x^{-\frac{3}{2}}\,dx = -4x^{-1} - 4x^{-\frac{1}{2}} + c$

10 The derivatives of $f(x)$ and $g(x)$ are the same so the derivative of $f(x) - g(x)$ is 0.

This means that $f(x) - g(x)$ is $\int 0\,dx$ = a constant.

Exercise 9.1B — page 247

1 a $y = \int 4x - 2\,dx$

$y = 2x^2 - 2x + c$

b When $x = 0$, $y = 3$

$3 = 2 \times 0^2 - 2 \times 0 + c$

$3 = c$

The equation is $y = 2x^2 - 2x + 3$.

2 a $y = \int x^{\frac{1}{2}}\,dx$

$y = \frac{2}{3}x^{\frac{3}{2}} + c$

b When $x = 9$, $y = 25$

$25 = \frac{2}{3} \times 9^{\frac{3}{2}} + c$

$25 = \frac{2}{3} \times 27 + c$

$25 = 18 + c$

$c = 7$

The equation is $y = \frac{2}{3}x^{\frac{3}{2}} + 7$.

3 $y = \int 0.4x + 3\,dx$

$y = 0.2x^2 + 3x + c$

a $0 = 0 + c$ so the equation is $y = 0.2x^2 + 3x$

b $5 = c$

$y = 0.2x^2 + 3x + 5$

c $0 = 0.2 \times 5^2 + 3 \times 5 + c$

$c = -20$

$y = 0.2x^2 + 3x - 20$

4 $y = \int \frac{x^2 + 10}{x^2}\,dx = \int 1 + 10x^{-2}\,dx$

$= x - 10x^{-1} + c$

$2 = 5 - \frac{10}{5} + c$

$c = -1$

The equation is $y = x - \frac{10}{x} - 1$.

5 a If $x \neq 0$, then x^2 is positive. This means that $\frac{2}{x^2}$ is always positive.

b $y = \frac{2}{x^2}$

$y = \int \frac{2}{x^2}\,dx = \int 2x^{-2}\,dx$

$y = -2x^{-1} + c$

$4 = -1 + c$

$c = 5$

The equation is $y = -\frac{2}{x} + 5$.

6 a $y = \int 3x^2 - 3\,dx$

$= x^3 - 3x + c$

$2 = c$

The curve is $y = x^3 - 3x + 2$.

b The turning points are where $3x^2 - 3 = 0$.

$x^2 = 1$

$x = 1$ or -1

There are 2 turning points.

When $x = 1$, $y = 1 - 3 + 2 = 0$ so one point is $(1, 0)$.

When $x = -1$, $y = -1 + 3 + 2 = 4$ so the other point is $(-1, 4)$.

7 a $f(x) = \int \frac{20}{x^3}\,dx = \int 20x^{-3}\,dx = -10x^{-2} + c$

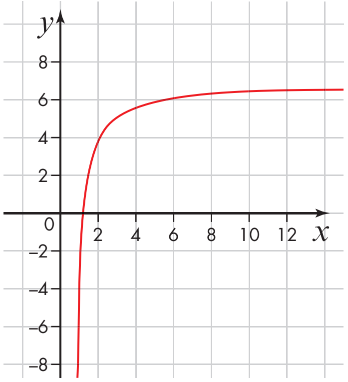
$f(2) = 4$

$4 = -\frac{10}{4} + c$

$c = 4 + 2.5 = 6.5$

$f(x) = -10x^{-2} + 6.5$ or $y = 6.5 - \frac{10}{x^2}$

b



8 a $y = \int 0.03x^2(x - 10)^2\,dx = \int 0.03x^2(x^2 - 20x + 100)$

$= \int 0.03x^4 - 0.6x^3 + 3x^2\,dx = 0.006x^5 - 0.15x^4 + x^3 + c$

When $x = 0$, $y = 100$, so $c = 100$

$y = 0.006x^5 - 0.15x^4 + x^3 + 100$

b After 10 seconds, $x = 10$ and

$y = 0.006 \times 10^5 - 0.15 \times 10^4 + 10^3 + 100$

$= 600 - 1500 + 1000 + 100 = 200$

The distance from A is 200 m.

9 $y = \int 3x^2 - 12x + 8\,dx = x^3 - 6x^2 + 8x + c$

When $x = 0$, $y = 0$ so $c = 0$

$y = x^3 - 6x^2 + 8x$

Where the curve crosses the x-axis, $x^3 - 6x^2 + 8x = 0$.

$x(x^2 - 6x + 8) = 0$

$x(x - 2)(x - 4) = 0$

$x = 0, 2$ or 4

The curve crosses the axes at $(0, 0)$, $(2, 0)$ and $(4, 0)$.

10 a $y = \int 4x^{-\frac{2}{3}}\,dx = 4 \times 3x^{\frac{1}{3}} + c$

$= 12x^{\frac{1}{3}} + c$

When $x = 8$, $y = 30$

$30 = 12 \times 8^{\frac{1}{3}} + c$

$30 = 12 \times 2 + c$

$c = 6$

$y = 12x^{\frac{1}{3}} + 6$

b When $x = 20$, $y = 12 \times 20^{\frac{1}{3}} + 6 = 38.57\ldots$

The radius is 38.6 cm to 3 s.f.

Exercise 9.2A page 255

1 a $\int_1^2 \frac{1}{3}x^2\,dx = \left[\frac{1}{9}x^3\right]_1^2 = \frac{8}{9} - \frac{1}{9} = \frac{7}{9}$

b $\left[\frac{1}{9}x^3\right]_2^4 = \frac{64}{9} - \frac{8}{9} = \frac{56}{9}$

c $\left[\frac{1}{9}x^3\right]_{-1}^3 = \left[\frac{27}{9}\right] - \left[-\frac{1}{9}\right] = \frac{28}{9}$

2 a $\left[\frac{1}{5}x^5\right]_1^2 = \left[\frac{32}{5}\right] - \left[\frac{1}{5}\right] = \frac{31}{5} = 6.2$

b $\left[2.5x^4\right]_2^4 = [640] - [40] = 600$

c $\int_1^5 10x^{-2}\,dx = \left[-10x^{-1}\right]_1^5$

$= \left[-\frac{10}{5}\right] - [-10] = -2 + 10 = 8$

3 **a** $\left[2x^3-2x^2\right]_3^4=[128-32]-[54-18]=96-36=60$

b $\left[x^3+2x^2+3x\right]_0^2=[8+8+6]-[0]=22$

c $\int_1^3 6x(x-1)\,dx=\int_1^3 6x^2-6x\,dx=\left[2x^3-3x^2\right.$

$=[54-27]-[2-3]=28$

4 **a** $\int_1^4 2x^{\frac{1}{2}}\,dx=\left[\frac{4}{3}x^{\frac{3}{2}}\right]_1^4=\left[\frac{4}{3}\times 8\right]-\left[\frac{4}{3}\right]=\frac{28}{3}$ or $9\frac{1}{3}$

b $\int_1^4 20x^{-\frac{1}{2}}\,dx=\left[40x^{\frac{1}{2}}\right]_1^4=[80]-[40]=40$

c $\int_1^4 8x^{\frac{1}{3}}\,dx=\left[6x^{\frac{4}{3}}\right]_1^4=\left[6\times 4^{\frac{4}{3}}\right]-[6$

$=32.10$ to 4 s.f.

5 **a** $\int_0^2 x^2-2x\,dx=\left[\frac{1}{3}x^3-x^2\right]_0^2=\left[\frac{8}{3}-4\right]-[0]=-\frac{4}{3}$

The area is $\frac{4}{3}$.

b Area $=\int_2^3 x^2-2x\,dx=\left[\frac{1}{3}x^3-x^2\right]$

$=[9-9]-\left[\frac{8}{3}-4\right]=0-\left[-\frac{4}{3}\right]=\frac{4}{3}$

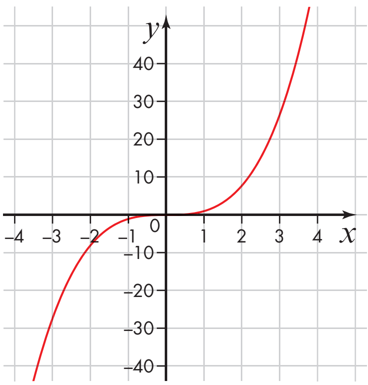
c Since the curve is symmetrical, this area is also $\frac{4}{3}$, the same as **part b**.

6 **a** **i** $\int_2^3 x^3\,dx=\left[\frac{1}{4}x^4\right]_2^3=\left[\frac{81}{4}\right]-[4]=16\frac{1}{4}$

ii $\int_{-2}^3 x^3\,dx=\left[\frac{1}{4}x^4\right]_{-2}^3=\left[\frac{81}{4}\right]-[4]=16\frac{1}{4}$

iii $\int_{-2}^2 x^3\,dx=\left[\frac{1}{4}x^4\right]_{-2}^2=[4]-[4]=0$

b



The integral from –2 to 2 in **part iii** above is zero because the graph is symmetrical and the areas above and below the x-axis are the same.

Also, $\int_{-2}^3 x^3\,dx = \int_{-2}^2 x^3\,dx + \int_2^3 x^3\,dx$

7 Shaded area $=\int_0^9 x^{\frac{1}{2}}\,dx=\left[\frac{2}{3}x^{\frac{3}{2}}\right]_0^9=\left[\frac{2}{3}\times 27\right]-[0]=18$

Area of OABC $=3\times 9=27$

18 is $\frac{2}{3}$ of 27

8 Area $=\int_{-2}^2 2+0.1x^4\,dx=\left[2x+0.02x^5\right]_{-2}^2$

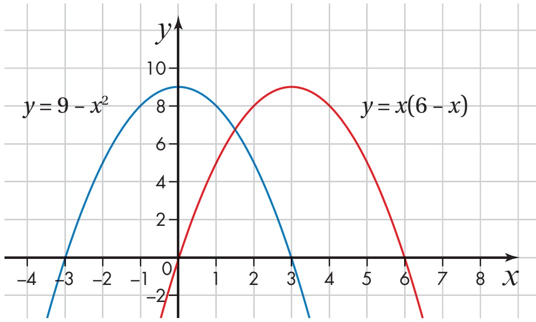
$=[4.64]-[-4.64]=9.28$

9 **a** $\int_{-3}^3 9-x^2\,dx=\left[9x-\frac{1}{3}x^3\right]_{-3}^3$

$=[27-9]-[-27+9]=36$

b $\int_0^6 6x-x^2\,dx=\left[3x^2-\frac{1}{3}x^3\right]_0^6=[108-72]-[0]=36$

c



One graph is a translation of the other and the areas are the same.

10 **a** **i** $\int_0^1 x^2\,dx=\left[\frac{1}{3}x^3\right]_0^1=\frac{1}{3}$

ii $\int_0^1 x^3\,dx=\left[\frac{1}{4}x^4\right]_0^1=\frac{1}{4}$

iii $\int_0^1 x^6\,dx=\left[\frac{1}{7}x^7\right]_0^1=\frac{1}{7}$

b $\int_0^1 x^n\,dx=\left[\frac{1}{n+1}x^{n+1}\right]_0^1=\frac{1}{n+1}$ when n is a positive integer.

11 **a** $\int_0^3 x^2(6-x)\,dx=\int_0^3 6x^2-x^3\,dx$

$=\left[2x^3-\frac{1}{4}x^4\right]_0^3=\left[54-20\frac{1}{4}\right]=33\frac{3}{4}$

Distance in first 3 seconds = 33.75 m

b $\int_3^4 6x^2-x^3\,dx=\left[2x^3-\frac{1}{4}x^4\right]_3^4$

$=[128-64]-\left[33\frac{3}{4}\right]=64-33\frac{3}{4}=30\frac{1}{4}$

Distance in fourth second = 30.25 m

12 Area $=\int_0^4 8+8x^{\frac{1}{2}}-6x\,dx=\left[8x+\frac{16}{3}x^{\frac{3}{2}}-3x^2\right]_0^4$

$=\left[32+42\frac{2}{3}-48\right]=26\frac{2}{3}$

13 a Area $= \int_3^5 30x^{-2}\,dx = \left[-30x^{-1}\right]_3^5$

$= \left[-\frac{30}{5}\right] - \left[-\frac{30}{3}\right] = -6 + 10 = 4$

b Area $= \int_3^{10} 30x^{-2}\,dx = \left[-30x^{-1}\right]_3^{10}$

$= \left[-\frac{30}{10}\right] - \left[-\frac{30}{3}\right] = -3 + 10 = 7$

c Area $= \int_3^n 30x^{-2}\,dx = \left[-30x^{-1}\right]_3^n$

$= \left[-\frac{30}{n}\right] - \left[-\frac{30}{3}\right] = -\frac{30}{n} + 10$ or $10 - \frac{30}{n}$

d $\frac{30}{n}$ is always positive for any positive value of n so $10 - \frac{30}{n}$ is always less than 10.

14 a Where the lines cross, $4 - x = x(4 - x)$

$4 - x = 4x - x^2$

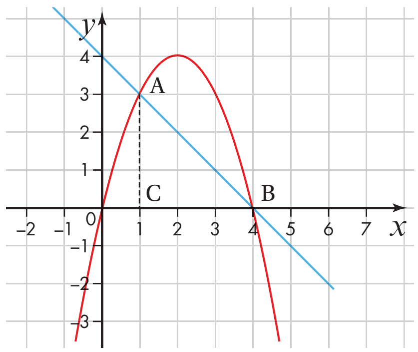
$x^2 - 5x + 4 = 0$

Factorise: $(x - 1)(x - 4) = 0$

$x = 1$ or 4

Points are (1, 3) and (4, 0).

b



The area required is the difference between the area under the curve between A and B, and the area of triangle ABC.

Area under curve $= \int_1^4 4x - x^2\,dx = \left[2x^2 - \frac{1}{3}x^3\right]_1^4$

$= \left[32 - \frac{64}{3}\right] - \left[2 - \frac{1}{3}\right]$

$= \left[10\frac{2}{3}\right] - \left[1\frac{2}{3}\right] = 9$

Area of triangle $= \frac{1}{2} \times 3 \times 3 = 4\frac{1}{2}$

Area between the curve and the straight line $= 9 - 4\frac{1}{2} = 4\frac{1}{2}$

Exam-style questions 9 page 258

1 $\int_1^3 8x^3 - 12x^2\,dx = 2x^4 - 4x^3 + c$

2 Distance $= \int_0^{10} 15 - 0.1x^2\,dx = \left[15x - \frac{0.1}{3}x^3\right]_0^{10}$

$= \left[150 - 33\frac{1}{3}\right] - [0] = 116\frac{2}{3}$ m

3 $\int_1^3 5x^{-2} + x^{-\frac{3}{2}}\,dx = -5x^{-1} - 2x^{-\frac{1}{2}} + c$

4 $y = \int -16x^{-2}\,dx = 16x^{-1} + c$

$= \frac{16}{x} + c$

When $x = 4$, $y = 6$

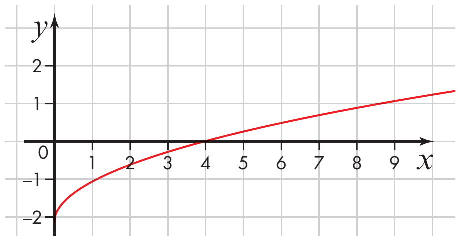
$6 = \frac{16}{4} + c$

$c = 6 - 4 = 2$

The equation is $y = \frac{16}{x} + 2$.

5 a Find some points on the curve.

x	0	1	4	9
y	–2	–1	0	1



b $\int_0^9 x^{\frac{1}{2}} - 2\,dx = \left[\frac{2}{3}x^{\frac{3}{2}} - 2x\right]_0^9 = [18 - 18] - [0] = 0$

c The area below the x-axis between 0 and 4 is the same as the area above it between 4 and 9.

6 a $y = \int 0.6x - 10\,dx = 0.3x^2 - 10x + c$

When $x = 0$, $y = 100$ so $c = 100$

$y = 0.3x^2 - 10x + 100$

When $x = 9$, $y = 0.3 \times 9^2 - 10 \times 9 + 100 = 34.3$

b When $x = 20$, $\frac{dy}{dx} = 0.6 \times 20 - 10 = 2$

This is positive and implies that the volume is increasing. This cannot be the case. If there is a leak, $\frac{dy}{dx}$ will be negative.

7 a You cannot integrate a fraction by integrating the numerator and the denominator separately and dividing one result by the other.

b Integral $= \int \frac{5}{4}x - 2x^{-2}\,dx = \frac{5}{8}x^2 + 2x^{-1} + c$

8 $y = \int 2x + 6\,dx$

$y = x^2 + 6x + c$

At a turning point, $\frac{dy}{dx} = 0$.

$2x + 6 = 0$

$x = -3$

The turning point is on the x-axis so the coordinates are (–3, 0).

Put these into the equation for y.

$0 = 9 - 18 + c$

$c = 9$

The equation of the curve is $y = x^2 + 6x + 9$.

9 Where the curve crosses the x-axis, $x(4-x)=0$.

$x=0$ or 4

Area $= \int_0^4 4kx - kx^2\,dx = \left[2kx^2 - \frac{1}{3}kx^3\right]_0^4$

$= \left[32k - 21\frac{1}{3}k\right] = 10\frac{2}{3}k$

$10\frac{2}{3}k = 32$

$k = 3$

10 Area beneath the curve between A and B is

$\int_{-a}^{a} x^2\,dx = \left[\frac{1}{3}x^3\right]_{-a}^{a} = \left[\frac{1}{3}a^3\right] - \left[-\frac{1}{3}a^3\right] = \frac{2}{3}a^3$

The area of rectangle ABCD $= 2a \times a^2 = 2a^3$

The area between AB and the curve is $2a^3 - \frac{2}{3}a^3$ $= \frac{4}{3}a^3$ and this is $\frac{2}{3}$ of the area of the rectangle.

11 First, find the coordinates of the points where the lines cross.

$x^2 - x + 4 = 2x + 8$

$x^2 - 3x - 4 = 0$

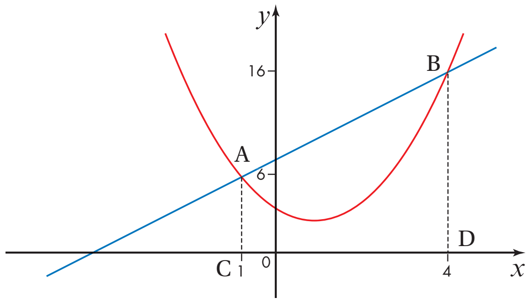
$(x-4)(x+1) = 0$

$x = -1$ or 4

When $x=-1$, $y = 1+1+4 = 6$

When $x = 4$, $y = 16 - 4 + 4 = 16$

The points are (–1, 6) and (4, 16).



You want the difference between the area of the trapezium ABCD and the area under the curve between A and B.

Area under curve $= \int_{-1}^{4} x^2 - x + 4\,dx$

$= \left[\frac{1}{3}x^3 - \frac{1}{2}x^2 + 4x\right]_{-1}^{4}$

$= \left[\frac{64}{3} - 8 + 16\right] - \left[-\frac{1}{3} - \frac{1}{2} - 4\right] = 29\frac{1}{3} + 4\frac{5}{6} = 34\frac{1}{6}$

Area of trapezium $= \frac{6+16}{2} \times 5 = 55$

The required area is $55 - 34\frac{1}{6} = 20\frac{5}{6}$.

12 Where they cross, $\frac{1}{16}x^3 = 2x^{\frac{1}{2}}$ and either $x = 0$ or $x^{\frac{5}{2}} = 32$ so $x = 4$

Area between $y = \frac{1}{16}x^3$ and the x-axis is

$\int_0^4 \frac{1}{16}x^3\,dx = \left[\frac{1}{64}x^4\right]_0^4 = 4 - 0 = 4$

Area between $y = 2\sqrt{x}$ and the x-axis is

$\int_0^4 2x^{\frac{1}{2}}\,dx = \left[\frac{4}{3}x^{\frac{3}{2}}\right]_0^4 = \frac{32}{3} - 0 = 10\frac{2}{3}$

Area between curves $= 10\frac{2}{3} - 4 = 6\frac{2}{3}$

10 Vectors

Prior knowledge — page 261

1 a $\sqrt{(6-1)^2 + (-10-2)^2} = 13$

b $\sqrt{(-5-17)^2 + (-4-7)^2} = 11\sqrt{5}$

2 $\tan^{-1}\left(\frac{5}{8}\right) = 32.0°$

3 $AB^2 = 10^2 + 17^2 - 2 \times 10 \times 17 \times \cos 142°$

$AB = \sqrt{656.9} = 25.6$ cm

Exercise 10.1A — page 265

1 a $\tan^{-1}\left(\frac{4}{7}\right) = 29.7°$

b $\tan^{-1}\left(\frac{7}{10}\right) = 35.0°$

$180 - 35.0 = 145.0°$

c $\tan^{-1}\left(\frac{5}{6}\right) = 39.8°$

$180 - 39.8 = 140.2°$

Angle $= -140.2°$

d $\tan^{-1}\left(\frac{9}{4}\right) = 66.0°$

Angle $= -66.0°$

2 a i Magnitude $= \sqrt{24^2 + 7^2} = 25$

Bearing $= 90 - \tan^{-1}\left(\frac{7}{24}\right) = 074°$

ii Magnitude $= \sqrt{(-12)^2 + (-35)^2} = 37$

Bearing $= 270 - \tan^{-1}\left(\frac{35}{12}\right) = 199°$

iii Magnitude $= \sqrt{182^2 + (-120)^2} = 218$

Bearing $= 90 + \tan^{-1}\left(\frac{120}{182}\right) = 123°$

b i Magnitude $= \sqrt{(-16)^2 + 8^2} = 8\sqrt{5}$

Bearing $= 270 + \tan^{-1}\left(\frac{8}{16}\right) = 297°$

ii Magnitude $= \sqrt{7^2 + 1^2} = 5\sqrt{2}$

Bearing $= 90 - \tan^{-1}\left(\frac{1}{7}\right) = 082°$

iii Magnitude $= \sqrt{2^2 + (-4)^2} = 2\sqrt{5}$

Bearing $= 90 + \tan^{-1}\left(\frac{4}{2}\right) = 153°$

3 a Both are correct. If a is positive, then the bearing is 045° but if a is negative then the bearing is 225°.

b $\begin{bmatrix} -a \\ a \end{bmatrix}$ (a > 0; same number but different signs)

4 11, 3 is possible because $11^2 + 3^2 = 130$. $9^2 + 7^2$ also equals 130. Apart from the negatives of these, there are no other integer pairs with this property.

Here are the 16 possible pairs of values:

11, 3	11, –3	–11, 3	–11, –3
3, 11	3, –11	–3, 11	–3, –11
9, 7	9, –7	–9, 7	–9, –7
7, 9	7, –9	–7, 9	–7, –9

5 a i $\sqrt{(-4)^2 + 3^2} = 5$

Unit vector $= \frac{1}{5}(-4\mathbf{i} + 3\mathbf{j})$

ii $\sqrt{143^2 + (-24)^2} = 145$

Unit vector $= \frac{1}{145}(143\mathbf{i} - 24\mathbf{j})$

b i $\sqrt{(-5)^2 + 12^2} = 13$

Unit vector $= \frac{1}{13}(-5\mathbf{i} + 12\mathbf{j})$

$52 \times \frac{1}{13}(-5\mathbf{i} + 12\mathbf{j}) = -20\mathbf{i} + 48\mathbf{j}$

ii $\sqrt{63^2 + 16^2} = 65$

Unit vector $= \frac{1}{65}(63\mathbf{i} + 16\mathbf{j})$

$195 \times \frac{1}{65}(63\mathbf{i} + 16\mathbf{j}) = 189\mathbf{i} + 48\mathbf{j}$

6 $\sqrt{144^2 + (-17)^2} = 145$

Unit vector $= \frac{1}{145}(144\mathbf{i} - 17\mathbf{j})$

$29 \times \frac{1}{145}(144\mathbf{i} - 17\mathbf{j}) = (28.8\mathbf{i} - 3.4\mathbf{j})\ \text{km h}^{-1}$

7 a $x = 7\sqrt{2}\cos 45° = 7$

$y = 7\sqrt{2}\sin 45° = 7$

For 135°, x is positive and y is negative.

$\begin{bmatrix} 7 \\ -7 \end{bmatrix}$

b $x = 2\sin 60° = \sqrt{3}$

$y = 2\cos 60° = 1$

For 060°, both x and y are positive.

$\begin{bmatrix} \sqrt{3} \\ 1 \end{bmatrix}$

c $x = 14\sin 30° = 7$

$y = 14\cos 30° = 7\sqrt{3}$

For 210°, both x and y are negative.

$\begin{bmatrix} -7 \\ -7\sqrt{3} \end{bmatrix}$

8 a From the cosine rule,

$a^2 = 14^2 + 9^2 - 2 \times 14 \times 9 \times \cos 135° = 455.19$

$a = \sqrt{455.19} = 21.3\ \text{km}$

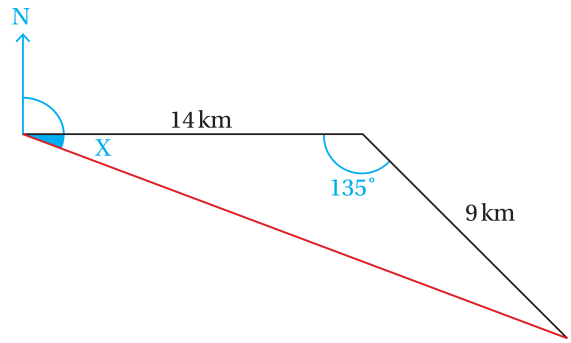
b $((14 + \frac{9}{2}\sqrt{2})\mathbf{i} - \frac{9}{2}\sqrt{2}\,\mathbf{j})\ \text{km}$

c $\frac{\sin X}{9} = \frac{\sin 135}{21.3}$

$\sin X = 0.298$

$X = 17.4°$

Bearing $= 90° + 17.4° = 107°$



Exercise 10.2A page 268

1 a i $\begin{bmatrix} 2 \\ 12 \end{bmatrix}$ **ii** $\begin{bmatrix} 7 \\ -4 \end{bmatrix}$ **iii** $\begin{bmatrix} 13 \\ -4 \end{bmatrix}$

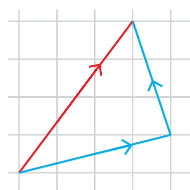
b i $\begin{bmatrix} -4 \\ -2 \end{bmatrix}$ **ii** $\begin{bmatrix} 0 \\ 10 \end{bmatrix}$ **iii** $\begin{bmatrix} -2 \\ -11 \end{bmatrix}$

c i $\begin{bmatrix} 10 \\ 20 \end{bmatrix}$ **ii** $\begin{bmatrix} 11 \\ 4.5 \end{bmatrix}$ **iii** $\begin{bmatrix} -20 \\ 12 \end{bmatrix}$

d i $\begin{bmatrix} 3 \\ -11 \end{bmatrix}$ **ii** $\begin{bmatrix} 1 \\ 18 \end{bmatrix}$ **iii** $\begin{bmatrix} 29 \\ -4.25 \end{bmatrix}$

2 a i $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$

ii It is the vector which joins the end of $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$ to the start of $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$.



b i $\begin{bmatrix} 5 \\ -2 \end{bmatrix}$

ii

iii $-\mathbf{b}$ is the opposite vector, so is parallel to (but in the opposite direction to $\mathbf{b}$). $\begin{bmatrix}5\\-2\end{bmatrix}$ is the vector which joins the end of $\begin{bmatrix}4\\1\end{bmatrix}$ to the start of $-\begin{bmatrix}-1\\3\end{bmatrix}$.

c **i** $\begin{bmatrix}8\\2\end{bmatrix}$ **ii**

iii It is the vector twice as long as **a** and parallel to **a**.

3 **a** $5p + 8q = 26$ ①

$10 - 10q = 15$ ②

From ②: $10q = -5$

$q = -\frac{1}{2}$

Substitute $q = -\frac{1}{2}$ into ①.

$5p - 4 = 26$

$5p = 30$

$p = 6$

b $2p + 4q = 10$ ①

$7p + 5q = 53$ ②

Multiply ① by 7.

$14p + 28q = 70$ ③

Multiply ② by 2.

$14p + 10q = 106$ ④

Subtract ④ from ③.

$18q = -36$

$q = -2$

Substitute $q = -2$ into ①.

$2p - 8 = 10$

$2p = 18$

$p = 9$

c $4p + 3q = 18$ ①

$-p + 5q = 7$ ②

Multiply ② by 4.

$-4p + 20q = 28$ ③

Add ① and ③.

$23q = 46$

$q = 2$

Substitute $q = 2$ into ①.

$4p + 6 = 18$

$4p = 12$

$p = 3$

4 **a** $p(4\mathbf{i} - \mathbf{j}) + 2(\mathbf{i} + 2\mathbf{j})\mathbf{b} = k(5\mathbf{i} + \mathbf{j})$

Equate **i** coefficients.

$4p + 2 = 5k$ ①

Equate **j** coefficients.

$-p + 4 = k$ ②

Substitute $k = -p + 4$ into ①.

$4p + 2 = 5(-p + 4)$

$4p + 2 = -5p + 20$

$9p = 18$

$p = 2$

b $3(4\mathbf{i} - \mathbf{j}) + q(\mathbf{i} + 2\mathbf{j}) = k(\mathbf{i} - \mathbf{j})$

Equate **i** coefficients.

$12 + q = k$ ①

Equate **j** coefficients.

$-3 + 2q = -k$ ②

Substitute $k = 12 + q$ into ②.

$-3 + 2q = -(12 + q)$

$-3 + 2q = -12 - q$

$3q = -9$

$q = -3$

5 **a** Equate **i** coefficients.

$2a + 8b = 6$ ①

Equate **j** coefficients.

$-a - 3b = -4$ ②

Multiply ② by 2.

$-2a - 6b = -8$ ③

Add ① and ③.

$2b = -2$

$b = -1$

Substitute $b = -1$ into ①.

$2a - 8 = 6$

$2a = 14$

$a = 7$

b If a vector is parallel to the x-axis then it has the form $k\begin{bmatrix}1\\0\end{bmatrix}$.

If a vector of the form $k\begin{bmatrix}1\\0\end{bmatrix}$ has a magnitude of 12, then $k = 12$.

Equate **j** coefficients.

$-a - 3b = 0$

$a = -3b$

Equate **i** coefficients.

$2a + 8b = 12$

Substitute $a = -3b$ into $2a + 8b = 12$.

$-6b + 8b = 12$

$2b = 12$

$b = 6$

$a = -3(6) = -18$

c If a vector is parallel to $y = -x$ then it has the form $k\begin{bmatrix}-1\\1\end{bmatrix}$.

Equate **i** coefficients.

$2a + 8b = -k$

Equate **j** coefficients.

$-a - 3b = k$

Substitute $k = -a - 3b$ into $2a + 8b = -k$.

$2a + 8b = -(-a - 3b)$

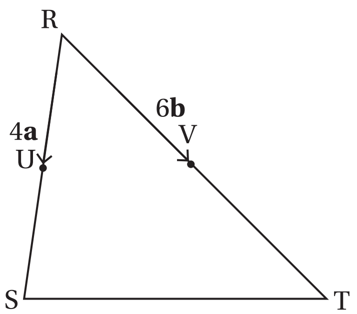
$2a + 8b = a + 3b$

$a = -5b$

Any pair of values for which $a = -5b$ will satisfy the requirement that the vector is parallel to $y = -x$.

Exercise 10.3A page 273

1 $\overrightarrow{ST} = \overrightarrow{SR} + \overrightarrow{RT} = -4\mathbf{a} + 6\mathbf{b} = 2(-2\mathbf{a} + 3\mathbf{b})$



$\overrightarrow{RU} = 2\mathbf{a}$

$\overrightarrow{RV} = 3\mathbf{b}$

$\overrightarrow{UV} = \overrightarrow{UR} + \overrightarrow{RV} = -2\mathbf{a} + 3\mathbf{b}$

Hence $\overrightarrow{ST} = 2\overrightarrow{UV}$.

If two vectors have a common factor then they are parallel.

2 a i $\overrightarrow{OB} = 6\mathbf{a} + 6\mathbf{c}$ **ii** $\overrightarrow{AC} = -6\mathbf{a} + 6\mathbf{c}$

iii $\overrightarrow{OU} = 6\mathbf{a} + 4\mathbf{c}$ **iv** $\overrightarrow{TA} = 3\mathbf{a} - 6\mathbf{c}$

v $\overrightarrow{OS} = 3\mathbf{a} + 3\mathbf{c}$ **vi** $\overrightarrow{US} = -3\mathbf{a} - \mathbf{c}$

vii $\overrightarrow{UT} = -3\mathbf{a} + 2\mathbf{c}$ **viii** $\overrightarrow{ST} = 3\mathbf{c}$

b The vectors are parallel.

$\overrightarrow{ST} = 3\mathbf{c}$ and $\overrightarrow{AB} = 6\mathbf{c}$, so $\overrightarrow{AB} = 2\overrightarrow{ST}$.

3 Ratio theorem:

$\overrightarrow{OX} = \frac{1}{13}(9 \times 5\mathbf{a} + 4 \times 6\mathbf{b})$

$= \frac{1}{13}(45\mathbf{a} + 24\mathbf{b}) = \frac{3}{13}(15\mathbf{a} + 8\mathbf{b})$

4 a $\overrightarrow{EF} = \overrightarrow{ED} + \overrightarrow{DF} = -12\mathbf{a} + 9\mathbf{b} = 3(-4\mathbf{a} + 3\mathbf{b})$

$\overrightarrow{MN} = \overrightarrow{MD} + \overrightarrow{DN} = -4\mathbf{a} + 3\mathbf{b}$

Since $\overrightarrow{EF}$ is parallel to $\overrightarrow{MN}$, FEMN is a trapezium.

b The ratio 2 : 1 means that the scale factor DM : DE = 1 : 3 and so the area of DEF is 9 times the area of DMN.

If the area of DEF is 72 units^2, then the area of DMN is 8 units^2 and the area of FEMN is 64 units^2.

5 $(10\mathbf{i} - \mathbf{j})$

6 $\overrightarrow{OM} = \frac{1}{3}\overrightarrow{OA} = 2\mathbf{a}$

$\overrightarrow{MB} = \overrightarrow{MO} + \overrightarrow{OB} = -2\mathbf{a} + 6\mathbf{b}$

$\overrightarrow{MX} = \lambda(-2\mathbf{a} + 6\mathbf{b})$

$\overrightarrow{OX} = \overrightarrow{OM} + \overrightarrow{MX} = 2\mathbf{a} + \lambda(-2\mathbf{a} + 6\mathbf{b})$

$= (2 - 2\lambda)\mathbf{a} + 6\lambda\mathbf{b}$

$\overrightarrow{ON} = \frac{1}{3}\overrightarrow{OB} = 2\mathbf{b}$

$\overrightarrow{NA} = \overrightarrow{NO} + \overrightarrow{OA} = -2\mathbf{b} + 6\mathbf{a}$

$\overrightarrow{NX} = \mu(-2\mathbf{b} + 6\mathbf{a})$

$\overrightarrow{OX} = \overrightarrow{ON} + \overrightarrow{NX} = 2\mathbf{b} + \mu(-2\mathbf{b} + 6\mathbf{a})$

$= (2 - 2\mu)\mathbf{b} + 6\mu\mathbf{a}$

Coefficients of **a**: $2 - 2\lambda = 6\mu$, so $3\mu + \lambda = 1$ ①

Coefficients of **b**: $2 - 2\mu = 6\lambda$, so $\mu = 1 - 3\lambda$ ②

Substitute for μ from ② in ①.

$3(1 - 3\lambda) + \lambda = 1$

$3 - 9\lambda + \lambda = 1$

$2 = 8\lambda$

$\lambda = \frac{1}{4}$

$\mu = 1 - 3 \times \frac{1}{4} = \frac{1}{4}$

Hence $\overrightarrow{OX} = (2 - 2 \times \frac{1}{4})\mathbf{a} + (6 \times \frac{1}{4})\mathbf{b} = \frac{3}{2}(\mathbf{a} + \mathbf{b})$

7 a $\overrightarrow{OM} = \frac{2}{3}\overrightarrow{OA} = \frac{2}{3}\mathbf{a}$

$\overrightarrow{MB} = \overrightarrow{MO} + \overrightarrow{OB} = -\frac{2}{3}\mathbf{a} + \mathbf{b}$

$\overrightarrow{MX} = \lambda(-\frac{2}{3}\mathbf{a} + \mathbf{b})$

$\overrightarrow{OX} = \overrightarrow{OM} + \overrightarrow{MX} = \frac{2}{3}\mathbf{a} + \lambda(-\frac{2}{3}\mathbf{a} + \mathbf{b})$

$= (\frac{2}{3} - \frac{2}{3}\lambda)\mathbf{a} + \lambda\mathbf{b}$

$\overrightarrow{ON} = \frac{2}{3}\overrightarrow{OB} = \frac{2}{3}\mathbf{b}$

$\overrightarrow{NA} = \overrightarrow{NO} + \overrightarrow{OA} = -\frac{2}{3}\mathbf{b} + \mathbf{a}$

$\overrightarrow{NX} = \mu(-\frac{2}{3}\mathbf{b} + \mathbf{a})$

$\overrightarrow{OX} = \overrightarrow{ON} + \overrightarrow{NX} = \frac{2}{3}\mathbf{b} + \mu(-\frac{2}{3}\mathbf{b} + \mathbf{a})$

$= (\frac{2}{3} - \frac{2}{3}\mu)\mathbf{b} + \mu\mathbf{a}$

Coefficients of **a**: $\frac{2}{3} - \frac{2}{3}\lambda = \mu$, so $3\mu + 2\lambda = 2$ ①

Coefficients of **b**: $\frac{2}{3} - \frac{2}{3}\mu = \lambda$, so $2\mu + 3\lambda = 2$ ②

Put ① = ②.

$3\mu + 2\lambda = 2\mu + 3\lambda$

$\mu = \lambda$

Substitute for μ in ②.

$2\lambda + 3\lambda = 2$

$5\lambda = 2$

$\lambda = \frac{2}{5}, \mu = \frac{2}{5}$

Hence $\overrightarrow{OX} = (\frac{2}{3} - \frac{2}{3} \times \frac{2}{5})\mathbf{b} + \frac{2}{5}\mathbf{a} = \frac{2}{5}\mathbf{a} + \frac{2}{5}\mathbf{b}$

$= \frac{2}{5}(\mathbf{a} + \mathbf{b})$

b $\overrightarrow{OM} = \frac{p}{p+q}\overrightarrow{OA} = \frac{p}{p+q}\mathbf{a}$

$\overrightarrow{MB} = \overrightarrow{MO} + \overrightarrow{OB} = -\frac{p}{p+q}\mathbf{a} + \mathbf{b}$

$\overrightarrow{MX} = \lambda(-\frac{p}{p+q}\mathbf{a} + \mathbf{b})$

$\overrightarrow{OX} = \overrightarrow{OM} + \overrightarrow{MX} = \frac{p}{p+q}\mathbf{a} + \lambda(-\frac{p}{p+q}\mathbf{a} + \mathbf{b})$

$= (\frac{p}{p+q} - \frac{p}{p+q}\lambda)\mathbf{a} + \lambda\mathbf{b}$

$\overrightarrow{ON} = \frac{p}{p+q}\overrightarrow{OB} = \frac{p}{p+q}\mathbf{b}$

$\overrightarrow{NA} = \overrightarrow{NO} + \overrightarrow{OA} = -\frac{p}{p+q}\mathbf{b} + \mathbf{a}$

$\overrightarrow{NX} = \mu(-\frac{p}{p+q}\mathbf{b} + \mathbf{a})$

$\overrightarrow{OX} = \overrightarrow{ON} + \overrightarrow{NX} = \frac{p}{p+q}\mathbf{b} + \mu(-\frac{p}{p+q}\mathbf{b} + \mathbf{a})$

$= (\frac{p}{p+q} - \frac{p}{p+q}\mu)\mathbf{b} + \mu\mathbf{a}$

Coefficients of **a**: $\frac{p}{p+q} - \frac{p}{p+q}\lambda = \mu$, so $\mu + \frac{p}{p+q}\lambda = \frac{p}{p+q}$ ①

Coefficients of **b**: $\frac{p}{p+q} - \frac{p}{p+q}\mu = \lambda$, so $\frac{p}{p+q}\mu + \lambda = \frac{p}{p+q}$ ②

Put ① = ②:

$\mu + \frac{p}{p+q}\lambda = \frac{p}{p+q}\mu + \lambda$

$\frac{\mu(p+q)+p\lambda}{p+q} = \frac{p\mu+\lambda(p+q)}{p+q}$

$\mu(p+q) + p\lambda = p\mu + \lambda(p+q)$

$p\mu + q\mu + p\lambda = p\mu + p\lambda + q\lambda$

$q\mu = q\lambda$

$\mu = \lambda$

Substitute for μ in ①.

$\lambda + \frac{p}{p+q}\lambda = \frac{p}{p+q}$

$\frac{\lambda(p+q)+p\lambda}{p+q}\lambda = \frac{p}{p+q}$

$\lambda(p+q) + p\lambda = p$

$p\lambda + q\lambda + p\lambda = p$

$\lambda(2p+q) = p$

$\lambda = \frac{p}{2p+q}$

Hence $\overrightarrow{OX} = \left(\frac{p}{p+q} - \frac{p}{p+q} \times \frac{p}{2p+q}\right)\mathbf{a} + \frac{p}{2p+q}\mathbf{b}$

$= \left(\frac{p}{p+q} - \frac{p^2}{(p+q)(2p+q)}\right)\mathbf{a} + \frac{p}{2p+q}\mathbf{b}$

$= \left(\frac{p(2p+q)}{(p+q)(2p+q)} - \frac{p^2}{(p+q)(2p+q)}\right)\mathbf{a} + \frac{p}{2p+q}\mathbf{b}$

$= \left(\frac{p(2p+q)-p^2}{(p+q)(2p+q)}\right)\mathbf{a} + \frac{p}{2p+q}\mathbf{b}$

$= \left(\frac{2p^2+pq-p^2}{(p+q)(2p+q)}\right)\mathbf{a} + \frac{p}{2p+q}\mathbf{b}$

$= \left(\frac{p^2+pq}{(p+q)(2p+q)}\right)\mathbf{a} + \frac{p}{2p+q}\mathbf{b}$

$= \left(\frac{p(p+q)}{(p+q)(2p+q)}\right)\mathbf{a} + \frac{p}{2p+q}\mathbf{b}$

$= \left(\frac{p}{2p+q}\right)\mathbf{a} + \frac{p}{2p+q}\mathbf{b}$

$= \frac{p}{2p+q}(\mathbf{a} + \mathbf{b})$

8 **a** $\overrightarrow{JK}$ and $\overrightarrow{LM}$ are parallel.

b $\overrightarrow{JN} = 6\mathbf{b}$

$\overrightarrow{NM} = 9\mathbf{b}$

P could be dividing $\overrightarrow{NM}$ in the ratio 1 : 2, so $\overrightarrow{MP} = -6\mathbf{b}$

$\overrightarrow{LP} = \overrightarrow{LM} + \overrightarrow{MP} = 6\mathbf{a} - 6\mathbf{b}$

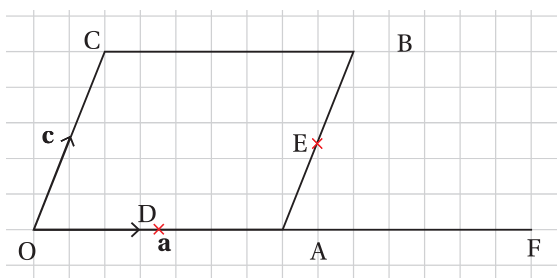
Alternatively, N could be dividing $\overrightarrow{MP}$ in the ratio 1 : 1, so $\overrightarrow{MP} = -18\mathbf{b}$

$\overrightarrow{LP} = \overrightarrow{LM} + \overrightarrow{MP} = 6\mathbf{a} - 18\mathbf{b}$

Exercise 10.3B

page 277

1



Since OABC is a parallelogram, $\overrightarrow{OA} = \overrightarrow{CB} = \mathbf{a}$ and $\overrightarrow{OC} = \overrightarrow{AB} = \mathbf{c}$.

a $\overrightarrow{OB} = \overrightarrow{OC} + \overrightarrow{CB} = \mathbf{c} + \mathbf{a}$

$\overrightarrow{DE} = \overrightarrow{DA} + \overrightarrow{AE} = \frac{1}{2}\overrightarrow{OA} + \frac{1}{2}\overrightarrow{AB} = \frac{1}{2}\mathbf{c} + \frac{1}{2}\mathbf{a}$

$= \frac{1}{2}(\mathbf{c} + \mathbf{a})$

So $\overrightarrow{DE} = \frac{1}{2}\overrightarrow{OB}$

If two vectors have a common factor then they are parallel.

b $\overrightarrow{CF} = \overrightarrow{CO} + \overrightarrow{OF} = \overrightarrow{CO} + 2\overrightarrow{OA} = -\mathbf{c} + 2\mathbf{a}$

$\overrightarrow{CE} = \overrightarrow{CB} + \overrightarrow{BE} = \overrightarrow{CB} + \frac{1}{2}\overrightarrow{BA} = \mathbf{a} + \frac{1}{2}(-\mathbf{c})$
$= -\frac{1}{2}\mathbf{c} + \mathbf{a} = \frac{1}{2}(-\mathbf{c} + 2\mathbf{a})$

So $\overrightarrow{CE} = \frac{1}{2}\overrightarrow{CF}$

If two vectors have a common factor then they are parallel.

If two vectors are parallel and include a common point, all three points are collinear.

2 $\overrightarrow{TU} = 3\mathbf{a} - 2\mathbf{c}$

$\overrightarrow{AX} = 4\mathbf{a}$

$\overrightarrow{TX} = 3\mathbf{a} - 6\mathbf{c} + 6\mathbf{a} = 9\mathbf{a} - 6\mathbf{c} = 3(3\mathbf{a} - 2\mathbf{c})$

$\overrightarrow{TU}$ and $\overrightarrow{TX}$ have a common factor, so are parallel.

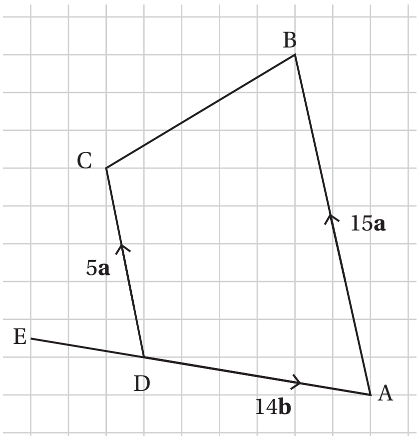
They also share a common point, so T, U and X are collinear.

3 $\overrightarrow{BC} = \overrightarrow{BA} + \overrightarrow{AD} + \overrightarrow{DC} = -15\mathbf{a} - 14\mathbf{b} + 5\mathbf{a}$
$= -10\mathbf{a} - 14\mathbf{b} = 2(-5\mathbf{a} - 7\mathbf{b})$

$\overrightarrow{BE} = \overrightarrow{BA} + \overrightarrow{AE} = \overrightarrow{BA} + 1.5\overrightarrow{AD}$
$= -15\mathbf{a} + 1.5(-14\mathbf{b}) = -15\mathbf{a} - 21\mathbf{b} = 3(-5\mathbf{a} - 7\mathbf{b})$

If two vectors have a common factor then they are parallel.

If two vectors are parallel and include a common point, all three points are collinear.



4 $\overrightarrow{JG} = 4\mathbf{b}$, $\overrightarrow{GF} = 12\mathbf{a}$, $\overrightarrow{GK} = 4\mathbf{a}$

$\overrightarrow{JK} = 4\mathbf{b} + 4\mathbf{a} = 4(\mathbf{b} + \mathbf{a})$

Since $\overrightarrow{JQ}$ is parallel to $\overrightarrow{JK}$, $\overrightarrow{JQ}$ can be given by $\overrightarrow{JQ} = n(\mathbf{b} + \mathbf{a})$.

$\overrightarrow{EF} = 8\mathbf{b}$

Since $\overrightarrow{FQ}$ is parallel to $\overrightarrow{EF}$, $\overrightarrow{FQ}$ can be given by $\overrightarrow{FQ} = k\mathbf{b}$.

$\overrightarrow{JQ} = \overrightarrow{JF} + \overrightarrow{FQ} = 4\mathbf{b} + 12\mathbf{a} + k\mathbf{b} = (k + 4)\mathbf{b} + 12\mathbf{a}$

Hence $(k + 4)\mathbf{b} + 12\mathbf{a} = n(\mathbf{b} + \mathbf{a})$.

$n = 12$

$k + 4 = 12$

$k = 8$

$\overrightarrow{DQ} = \overrightarrow{DE} + \overrightarrow{EF} + \overrightarrow{FQ} = 12\mathbf{a} + 8\mathbf{b} + 8\mathbf{b}$
$= 12\mathbf{a} + 16\mathbf{b}$

5 $\overrightarrow{PQ} = \overrightarrow{OR} = 4\mathbf{r}$

Since PQU is a straight line, $\overrightarrow{QU}$ is parallel to $\overrightarrow{PQ}$, so $\overrightarrow{QU} = k\mathbf{r}$.

$\overrightarrow{OS} = \frac{1}{2}(4\mathbf{r} + 6\mathbf{p}) = 2\mathbf{r} + 3\mathbf{p}$

$\overrightarrow{OT} = 4\mathbf{r} + 5\mathbf{p}$

$\overrightarrow{ST} = \overrightarrow{SO} + \overrightarrow{OT} = -(2\mathbf{r} + 3\mathbf{p}) + (4\mathbf{r} + 5\mathbf{p}) = 2\mathbf{r} + 2\mathbf{p}$
$= 2(\mathbf{r} + \mathbf{p})$

Since STU is a straight line, $\overrightarrow{TU}$ is parallel to $\overrightarrow{ST}$, so $\overrightarrow{TU} = n(\mathbf{r} + \mathbf{p})$.

$\overrightarrow{QU} = \overrightarrow{QT} + \overrightarrow{TU} = -\mathbf{p} + n(\mathbf{r} + \mathbf{p})$

Hence $k\mathbf{r} = -\mathbf{p} + n(\mathbf{r} + \mathbf{p})$

Equate coefficients for $\mathbf{p}$.

$-1 + n = 0$

$n = 1$

Equate coefficients for $\mathbf{r}$.

$k = n$

$k = 1$

$\overrightarrow{OU} = \overrightarrow{OP} + \overrightarrow{PQ} + \overrightarrow{QU} = 6\mathbf{p} + 4\mathbf{r} + \mathbf{r} = 6\mathbf{p} + 5\mathbf{r}$

Exercise 10.4A page 281

1 $\mathbf{r}_H - \mathbf{r}_S = (-\mathbf{i} + 7\mathbf{j}) - (5\mathbf{i} + 2\mathbf{j}) = (-6\mathbf{i} + 5\mathbf{j})$ km

$\mathbf{r}_H - \mathbf{r}_C = (-\mathbf{i} + 7\mathbf{j}) - (3\mathbf{i} - 9\mathbf{j}) = (-4\mathbf{i} + 16\mathbf{j})$ km

$\mathbf{r}_H - \mathbf{r}_M = (-\mathbf{i} + 7\mathbf{j}) - (-2\mathbf{i} - 4\mathbf{j}) = (\mathbf{i} + 11\mathbf{j})$ km

$\mathbf{r}_S - \mathbf{r}_C = (5\mathbf{i} + 2\mathbf{j}) - (3\mathbf{i} - 9\mathbf{j}) = (2\mathbf{i} + 11\mathbf{j})$ km

$\mathbf{r}_S - \mathbf{r}_M = (5\mathbf{i} + 2\mathbf{j}) - (-2\mathbf{i} - 4\mathbf{j}) = (7\mathbf{i} + 6\mathbf{j})$ km

$\mathbf{r}_C - \mathbf{r}_M = (3\mathbf{i} - 9\mathbf{j}) - (-2\mathbf{i} - 4\mathbf{j}) = (5\mathbf{i} - 5\mathbf{j})$ km

a **i** Supermarket and church

ii Supermarket and museum

iii Church and hospital

b $(\mathbf{i} + 11\mathbf{j})$ km

c Magnitude of $(5\mathbf{i} - 5\mathbf{j}) = \sqrt{5^2 + (-5)^2}$, so church and museum.

2 **a** $A = 4\mathbf{i} - 2\mathbf{j}$, $B = -\mathbf{i} + 3\mathbf{j}$, $C = -3\mathbf{i} - 3\mathbf{j}$

b $\overrightarrow{AB} = (-\mathbf{i} + 3\mathbf{j}) - (4\mathbf{i} - 2\mathbf{j}) = -5\mathbf{i} + 5\mathbf{j}$,

$\overrightarrow{AC} = (-3\mathbf{i} - 3\mathbf{j}) - (4\mathbf{i} - 2\mathbf{j}) = -7\mathbf{i} - \mathbf{j}$

$\overrightarrow{BC} = (-3\mathbf{i} - 3\mathbf{j}) - (-\mathbf{i} + 3\mathbf{j}) = -2\mathbf{i} - 6\mathbf{j}$

c $AB = \sqrt{(-5)^2 + 5^2} = 5\sqrt{2}$

$AC = \sqrt{(-7)^2 + (-1)^2} = 5\sqrt{2}$

$BC = \sqrt{(-2)^2 + (-6)^2} = 2\sqrt{10}$

d ABC is isosceles because two sides are exactly the same length.

3 a $\overrightarrow{ST} = (-2\mathbf{i} + 3\mathbf{j}) - (4\mathbf{i} - 6\mathbf{j}) = -6\mathbf{i} + 9\mathbf{j}$

$\sqrt{(-6)^2 + 9^2} = 3\sqrt{13}$

b $\overrightarrow{ST} = (-2\mathbf{i} + 3\mathbf{j}) - (q\mathbf{i} - 6\mathbf{j}) = (-2 - q)\mathbf{i} + 9\mathbf{j}$

$(-2 - q)^2 + 9^2 = 15^2$

$(-2 - q)^2 = 225 - 81 = 144$

$-2 - q = \pm 12$

$q = 10$ or -14

4 a $\overrightarrow{AB} = \mathbf{b} - \mathbf{a} = (8\mathbf{i} + 4\mathbf{j}) - (-\mathbf{i} - 4\mathbf{j}) = 9\mathbf{i} + 8\mathbf{j}$

$\overrightarrow{DC} = \mathbf{c} - \mathbf{d} = (3\mathbf{i} + 6\mathbf{j}) - (-6\mathbf{i} - 2\mathbf{j}) = 9\mathbf{i} + 8\mathbf{j}$

$\overrightarrow{BC} = \mathbf{c} - \mathbf{b} = (3\mathbf{i} + 6\mathbf{j}) - (8\mathbf{i} + 4\mathbf{j}) = -5\mathbf{i} + 2\mathbf{j}$

$\overrightarrow{AD} = \mathbf{d} - \mathbf{a} = (-6\mathbf{i} - 2\mathbf{j}) - (-\mathbf{i} - 4\mathbf{j}) = -5\mathbf{i} + 2\mathbf{j}$

This confirms that the shape has two pairs of parallel sides.

Also, since AB ≠ BC and the gradients of AB and BC are not perpendicular, the shape is a parallelogram.

b $\overrightarrow{CD} = \mathbf{d} - \mathbf{c} = (-6\mathbf{i} - 2\mathbf{j}) - (3\mathbf{i} + 6\mathbf{j}) = -9\mathbf{i} - 8\mathbf{j}$

$\overrightarrow{FE} = \mathbf{e} - \mathbf{f} = (-5\mathbf{i} - 14\mathbf{j}) - (4\mathbf{i} - 6\mathbf{j}) = -9\mathbf{i} - 8\mathbf{j}$

$\overrightarrow{DE} = \mathbf{e} - \mathbf{d} = (-5\mathbf{i} - 14\mathbf{j}) - (-6\mathbf{i} - 2\mathbf{j}) = \mathbf{i} - 12\mathbf{j}$

$\overrightarrow{CF} = \mathbf{f} - \mathbf{c} = (4\mathbf{i} - 6\mathbf{j}) - (3\mathbf{i} + 6\mathbf{j}) = \mathbf{i} - 12\mathbf{j}$

This confirms that the shape has two pairs of parallel sides.

Also, since CD = DE (because $9^2 + 8^2 = 1^2 + 12^2$), and the gradients of CD and DE are not perpendicular, the shape is a rhombus.

c $\overrightarrow{DF} = \mathbf{f} - \mathbf{d} = (4\mathbf{i} - 6\mathbf{j}) - (-6\mathbf{i} - 2\mathbf{j}) = 10\mathbf{i} - 4\mathbf{j}$

$\overrightarrow{CB} = \mathbf{b} - \mathbf{c} = (8\mathbf{i} + 4\mathbf{j}) - (3\mathbf{i} + 6\mathbf{j}) = 5\mathbf{i} - 2\mathbf{j}$

$\overrightarrow{FB} = \mathbf{b} - \mathbf{f} = (8\mathbf{i} + 4\mathbf{j}) - (4\mathbf{i} - 6\mathbf{j}) = 4\mathbf{i} + 10\mathbf{j}$

$\overrightarrow{DC} = \mathbf{c} - \mathbf{d} = (3\mathbf{i} + 6\mathbf{j}) - (-6\mathbf{i} - 2\mathbf{j}) = 9\mathbf{i} + 8\mathbf{j}$

There are no identical sides, but DF = 2CB.

This confirms that there is one pair of parallel sides. Hence the shape is a trapezium.

5 a Since PQRS is a square, $\overrightarrow{PS} = \overrightarrow{QR} = \mathbf{r} - \mathbf{q}$

$= (-2\mathbf{i} + \mathbf{j}) - (6\mathbf{i} + 7\mathbf{j}) = -8\mathbf{i} - 6\mathbf{j}$

$\overrightarrow{OS} = \overrightarrow{OP} + \overrightarrow{PS} = \overrightarrow{OP} + \overrightarrow{QR}$

$\overrightarrow{OS} = 12\mathbf{i} - \mathbf{j} - 8\mathbf{i} - 6\mathbf{j} = 4\mathbf{i} - 7\mathbf{j}$

b The centre of the square is halfway between opposite vertices such as P and R.

$\frac{1}{2}[(12\mathbf{i} - \mathbf{j}) + (-2\mathbf{i} + \mathbf{j})] = 5\mathbf{i}$

c Length of side = magnitude of $\overrightarrow{QR}$

$= \sqrt{(-8)^2 + (-6)^2} = 10$

Area = 10 × 10 = 100

6 $\overrightarrow{JK} = \mathbf{k} - \mathbf{j} = (9\mathbf{i} - 4\mathbf{j}) - (-3\mathbf{i} + \mathbf{j}) = 12\mathbf{i} - 5\mathbf{j}$

$\overrightarrow{JL} = \mathbf{l} - \mathbf{j} = (3\mathbf{i} + 5\mathbf{j}) - (-3\mathbf{i} + \mathbf{j}) = 6\mathbf{i} + 4\mathbf{j}$

$\overrightarrow{KL} = \mathbf{l} - \mathbf{k} = (3\mathbf{i} + 5\mathbf{j}) - (9\mathbf{i} - 4\mathbf{j}) = -6\mathbf{i} + 9\mathbf{j}$

Magnitude of $\overrightarrow{JK} = \sqrt{12^2 + (-5)^2} = 13$

Magnitude of $\overrightarrow{JL} = \sqrt{6^2 + 4^2} = \sqrt{52} = 2\sqrt{13}$

Magnitude of $\overrightarrow{KL} = \sqrt{(-6)^2 + 9^2} = \sqrt{117} = 3\sqrt{13}$

Since $(3\sqrt{13})^2 + (2\sqrt{13})^2 = 13^2$, the sides of the triangle obey Pythagoras' theorem.

Hence the triangle is right-angled.

Exam-style questions 10 — page 283

1 a Speed $= \sqrt{(-11)^2 + 21^2} = \sqrt{562} = 23.7\,\text{m s}^{-1}$

b $\tan^{-1}\left(\frac{21}{11}\right) = 62.4°$.

Bearing = 270 + 62.4 = 332°

2 14:20 p.m. to 17:05 p.m. is 2.75 hours.

$\sqrt{2^2 + 1^2} = \sqrt{5}$

$2.75 \times \sqrt{5} = 6.1\,\text{km}$

3 a $(p + 3)^2 + (-4)^2 = 8^2 + (p - 5)^2$

$p^2 + 6p + 9 + 16 = 64 + p^2 - 10p + 25$

$6p + 25 = -10p + 89$

$16p = 64$

$p = 4$

b $\tan^{-1}\left(\frac{4}{7}\right) - \tan^{-1}\left(\frac{1}{8}\right) = 22.6°$

4 a Ratio theorem: $\overrightarrow{OP} = \frac{1}{5}(3 \times 8\mathbf{a} + 2 \times 7\mathbf{c})$

$= \frac{1}{5}(24\mathbf{a} + 14\mathbf{c}) = \frac{2}{5}(12\mathbf{a} + 7\mathbf{c})$

b $\overrightarrow{OB} = \overrightarrow{OC} + \overrightarrow{CB} = 7\mathbf{c} + 12\mathbf{a}$

$\overrightarrow{OP} = \frac{2}{5}\overrightarrow{OB}$

$\overrightarrow{OP}$ and $\overrightarrow{OB}$ have a common factor, so are parallel.

They also share a common point, so O, P and B are collinear.

c Since $\overrightarrow{OP} = \frac{2}{5}\overrightarrow{OB}$, OP : PB = 2 : 3.

5 a i $\begin{bmatrix} 5 \\ -6 \end{bmatrix} - 2\begin{bmatrix} k \\ 1 \end{bmatrix} = \begin{bmatrix} 5 - 2k \\ -8 \end{bmatrix}$

ii $\sqrt{(5 - 2k)^2 + (-8)^2}$

b $\sqrt{(5 - 2k)^2 + (-8)^2} > 2\sqrt{17}$

Solve $\sqrt{(5 - 2k)^2 + (-8)^2} = 2\sqrt{17}$

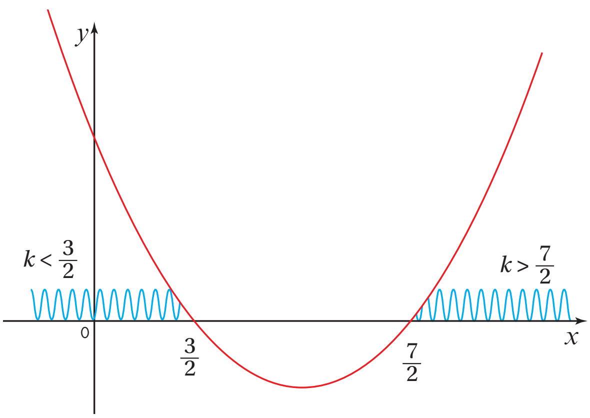
$(5 - 2k)^2 + 64 = 68$

$4k^2 - 20k + 25 + 64 = 68$

$4k^2 - 20k + 21 = 0$

$(2k - 3)(2k - 7) = 0$

$k = \frac{3}{2}$ or $\frac{7}{2}$



For magnitude $> 2\sqrt{17}$, $k < \frac{3}{2}$ or $k > \frac{7}{2}$.

6 All sides of a rhombus are equal in length. Hence AB and AD will be the same length.

$\overrightarrow{AB} = \mathbf{b} - \mathbf{a} = (2\mathbf{i} + 12\mathbf{j}) - (p\mathbf{i} + 5\mathbf{j}) = (2 - p)\mathbf{i} + 7\mathbf{j}$

$\overrightarrow{AD} = \mathbf{d} - \mathbf{a} = (22\mathbf{i} + 2\mathbf{j}) - (p\mathbf{i} + 5\mathbf{j}) = (22 - p)\mathbf{i} - 3\mathbf{j}$

Hence $(2 - p)^2 + 7^2 = (22 - p)^2 + (-3)^2$.

$p^2 - 4p + 4 + 49 = p^2 - 44p + 484 + 9$

$-4p + 53 = -44p + 493$

$40p = 440$

$p = 11$

Hence $\overrightarrow{AB} = (2 - 11)\mathbf{i} + 7\mathbf{j} = -9\mathbf{i} + 7\mathbf{j}$

$|\overrightarrow{AB}|^2 = (-9)^2 + 7^2 = 130$

Hence BC^2 will also equal 130.

$\overrightarrow{BC} = \mathbf{c} - \mathbf{b} = ((q + 4)\mathbf{i} + q\mathbf{j}) - (2\mathbf{i} + 12\mathbf{j})$

$= ((q + 2)\mathbf{i} + (q - 12)\mathbf{j})$

$(q + 2)^2 + (q - 12)^2 = 130$

$q^2 + 4q + 4 + q^2 - 24q + 144 = 130$

$2q^2 - 20q + 148 = 130$

$2q^2 - 20q + 18 = 0$

$q^2 - 10q + 9 = 0$

$(q - 1)(q - 9) = 0$

$q = 1$ or 9

Only one of these answers is valid. Note that CD^2 must also equal 130.

$\overrightarrow{CD} = \mathbf{d} - \mathbf{c} = (22\mathbf{i} + 2\mathbf{j}) - ((q + 4)\mathbf{i} + q\mathbf{j})$

$= (18 - q)\mathbf{i} + (2 - q)\mathbf{j}$

When $q = 1$, $\overrightarrow{CD} = 17\mathbf{i} + \mathbf{j}$

$17^2 + 1^2 \neq 130$

When $q = 9$, $\overrightarrow{CD} = 9\mathbf{i} - 7\mathbf{j}$

$9^2 + (-7)^2 = 130$

Hence $p = 11$ and $q = 9$

7 a Given three vertices of a rectangle, one must be a right angle.

Find the magnitude of each line segment.

$(-8\mathbf{i} - 4\mathbf{j}) - (-\mathbf{i} + 10\mathbf{j}) = (-7\mathbf{i} - 14\mathbf{j})$, magnitude

$= \sqrt{(-7)^2 + (-14)^2} = \sqrt{245}$

$(-8\mathbf{i} - 4\mathbf{j}) - (3\mathbf{i} + 8\mathbf{j}) = (-11\mathbf{i} - 12\mathbf{j})$, magnitude

$= \sqrt{(-11)^2 + (-12)^2} = \sqrt{265}$

$(-\mathbf{i} + 10\mathbf{j}) - (3\mathbf{i} + 8\mathbf{j}) = (-4\mathbf{i} + 2\mathbf{j})$, magnitude

$= \sqrt{(-4)^2 + 2^2} = \sqrt{20}$

Since $(\sqrt{20})^2 + (\sqrt{245})^2 = (\sqrt{265})^2$, the hypotenuse joins $(-8\mathbf{i} - 4\mathbf{j})$ and $(3\mathbf{i} + 8\mathbf{j})$, and the right angle is at $(-\mathbf{i} + 10\mathbf{j})$.

Area $= \sqrt{245} \times \sqrt{20} = 70$

b Fourth vertex $= (-8\mathbf{i} - 4\mathbf{j}) + [(3\mathbf{i} + 8\mathbf{j}) - (-\mathbf{i} + 10\mathbf{j})]$

$= \mathbf{(-4i - 6j)}$

8 Since ADP is a straight line, $\overrightarrow{DP}$ is parallel to $\overrightarrow{AD}$, so $\overrightarrow{DP} = k\mathbf{d}$

$\overrightarrow{XP} = \overrightarrow{XC} + \overrightarrow{CD} + \overrightarrow{DP} = 6\mathbf{d} - 10\mathbf{b} + k\mathbf{d}$

$\overrightarrow{XY} = 6\mathbf{d} - 4\mathbf{b}$

Also, $\overrightarrow{XP} = n\overrightarrow{XY} = n(6\mathbf{d} - 4\mathbf{b})$

Hence $6\mathbf{d} - 10\mathbf{b} + k\mathbf{d} = n(6\mathbf{d} - 4\mathbf{b})$

Equating coefficients for

b: $-10 = -4n$

$n = \frac{5}{2}$

Equating coefficients for

d: $6 + k = 6n$

$6 + k = 6 \times \frac{5}{2}$

$6 + k = 15$

$k = 9$

$\overrightarrow{BP} = \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DP} = 10\mathbf{d} - 10\mathbf{b} + 9\mathbf{d} = 19\mathbf{d} - 10\mathbf{b}$

9 a $\overrightarrow{CF} = \mathbf{f} - \mathbf{c} = (5\mathbf{i} + 4\mathbf{j}) - (-4\mathbf{i} + \mathbf{j}) = (9\mathbf{i} + 3\mathbf{j})$

$\sqrt{9^2 + 3^2} = \sqrt{90} = 3\sqrt{10}$

b $\overrightarrow{CD} = \mathbf{d} - \mathbf{c} = (-5\mathbf{i} - 6\mathbf{j}) - (-4\mathbf{i} + \mathbf{j}) = (-\mathbf{i} - 7\mathbf{j})$

$\overrightarrow{FE} = \mathbf{e} - \mathbf{f} = (4\mathbf{i} - 3\mathbf{j}) - (5\mathbf{i} + 4\mathbf{j}) = (-\mathbf{i} - 7\mathbf{j})$

c $\overrightarrow{DE} = \mathbf{e} - \mathbf{d} = (4\mathbf{i} - 3\mathbf{j}) - (-5\mathbf{i} - 6\mathbf{j}) = (9\mathbf{i} + 3\mathbf{j})$

$\overrightarrow{DX} = \frac{1}{3}\overrightarrow{DE} = \frac{1}{3}(9\mathbf{i} + 3\mathbf{j}) = (3\mathbf{i} + \mathbf{j})$

$\overrightarrow{OX} = \overrightarrow{OD} + \overrightarrow{DX} = (-5\mathbf{i} - 6\mathbf{j}) + (3\mathbf{i} + \mathbf{j}) = (-2\mathbf{i} - 5\mathbf{j})$

d $XC^2 = (-2 - (-4))^2 + (-5 - 1)^2 = 40$

$XD^2 = (-2 - (-5))^2 + (-5 - (-6))^2 = 10$

$CD^2 = (-4 - (-5))^2 + (1 - (-6))^2 = 50$

Since $XD^2 + XC^2 = CD^2$, XC is perpendicular to XD.

e Magnitude of $\overrightarrow{XC} = \sqrt{40} = 2\sqrt{10}$

Area $= 3\sqrt{10} \times 2\sqrt{10} = 60$

f $\overrightarrow{CG} = \mathbf{g} - \mathbf{c} = (2\mathbf{i} + 3\mathbf{j}) - (-4\mathbf{i} + \mathbf{j}) = (6\mathbf{i} + 2\mathbf{j}) = 2(3\mathbf{i} + \mathbf{j})$

Since $\overrightarrow{CG}$ and $\overrightarrow{DX}$ have a common factor, they are parallel and DXGC is a trapezium.

11 Proof

Prior knowledge page 286

1 Let $2n + 1$ be an odd number. Then $2n + 3$ and $2n + 5$ are consecutive odd numbers.

$2n + 1 + 2n + 3 + 2n + 5 = 6n + 9$

$= 3(2n + 3)$

which is a multiple of 3.

2 Let $2n$ be an even number and $2n + 1$ be the next number.

$2n + 2n + 1 = 4n + 1$ which is an odd number.

3 Let $2n$ be an even number and $2n + 1$ be the next number.

$(2n + 1)^2 - (2n)^2 = 4n^2 + 4n + 1 - 4n^2$

$= 4n + 1$

$= (2n) + (2n + 1)$

4 Let $2n$ be an even number and $2n + 2$ be the next even number.

$(2n)^2 + (2n + 2)^2 = 4n^2 + 4n^2 + 8n + 4$

$= 8n^2 + 8n + 4$

$= 4(2n^2 + 2n + 1)$

which is a multiple of 4.

5 $(2n - 1)^2 - (2n + 1)^2 = 4n^2 - 4n + 1 - (4n^2 + 4n + 1)$

$= -8n$

which is a multiple of 8.

Exercise 11.1A page 288

1 $n^3 - n = n(n^2 - 1) = n(n - 1)(n + 1)$

$(n - 1)$, n and $(n + 1)$ are three consecutive integers.

When $n = 2$, the integers are 1, 2 and 3.

When $n > 2$, the integers are three consecutive positive integers greater than 1, 2 and 3.

2 The shape is a rectangle with a rectangular part removed.

The area of the rectangle is

$(3x - 1)(2x + 5) = 6x^2 + 13x - 5$

The removed part has an area of

$((3x - 1) - (x + 3))((2x + 5) - x)$

$= (2x - 4)(x + 5)$

$= 2x^2 + 6x - 20$

The area of the shape is

$6x^2 + 13x - 5 - (2x^2 + 6x - 20) = 4x^2 + 7x + 15$

Hence $4x^2 + 7x + 15 = 201$

$4x^2 + 7x - 186 = 0$

$(4x + 31)(x - 6) = 0$

$x = 6$

Perimeter $= 2((3x - 1) + (2x + 5))$

$= 2((3(6) - 1) + (2(6) + 5)) = 2(17 + 17)$

$= 68$ cm

3 $n^2 - 6n + 10 = (n - 3)^2 - 9 + 10$

$= (n - 3)^2 + 1$

which is positive for all values of n.

4 Every even number can be expressed as $2n$.

$(2n)^2 = 4n^2$ which is a multiple of 4.

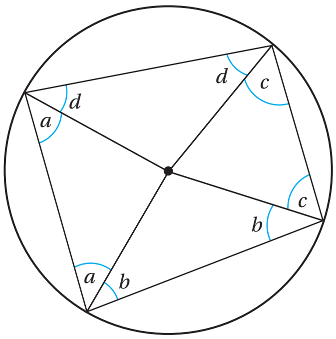
Every odd number can be expressed as $2n + 1$.

$(2n + 1)^2 = 4n^2 + 4n + 1$

$= 4(n^2 + n) + 1$

which is one more than a multiple of 4.

5 Join the four corners of the cyclic quadrilateral to the centre of the circle. This will create four isosceles triangles. Label the base angles of the four isosceles triangles as shown in the diagram.



$a + a + b + b + c + c + d + d = 360°$

$2a + 2b + 2c + 2d = 360°$

$a + b + c + d = 180°$

One pair of opposite angles is $(a + d)$ and $(b + c)$.

The other pair of opposite angles is $(a + b)$ and $(c + d)$.

Hence the opposite angles of a cyclic quadrilateral have a sum of 180°.

6 Let ABCDEF be a seating arrangement with A to F sitting in alphabetical order and with F next to A.

Start with A and B next to each other.

For ABC, E must come next. Hence there are two possibilities starting ABC: ABCEDF and ABCEFD.

For ABD, C must not come next.

For ABDE, C must come next. Hence there is one possibility starting ABDE: ABDECF.

For ABDF, there are two possibilities: ABDFCE and ABDFEC.

For ABE, C must come next. D cannot follow C. Hence there is one possibility starting ABE: ABECFD.

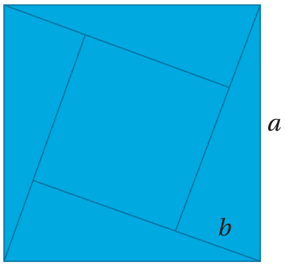
For ABFC, E must come next. Hence there is one possibility starting ABFC: ABFCED.

For ABFD, C must not come next. Hence there is one possibility starting ABFD: ABFDEC.

For ABFE, C must come next but would have to be followed by D, so this is not a possibility.

Therefore there are eight possible seating arrangements.

7



Area of large square $= a^2$

Height of one triangle $= \sqrt{a^2 - b^2}$

Area of one triangle $= \frac{b}{2}(\sqrt{a^2 - b^2})$

Length of small square $= \sqrt{a^2 - b^2} - b$

Area of small square $= (\sqrt{a^2 - b^2} - b)(\sqrt{a^2 - b^2} - b)$

Area of four triangles plus small square

$= \frac{4b}{2}(\sqrt{a^2 - b^2}) + (\sqrt{a^2 - b^2} - b)(\sqrt{a^2 - b^2} - b)$

$= 2b(\sqrt{a^2 - b^2}) + a^2 - b^2 - 2b\sqrt{a^2 - b^2} + b^2$

$= a^2$

8 The flaw is between the following two stages:

Factorising $a(b - a) = (b + a)(b - a)$

Dividing leaves $a = b + a$

If $b = a$ then $b - a = 0$ and you cannot divide by zero.

Exercise 11.2A — page 293

1 Case 1, ABC: A is next to B so this is allowed.

Case 2, ACB: A is not next to B so this is not allowed.

Case 3, BAC: A is next to B so this is allowed.

Case 4, BCA: A is not next to B so this is not allowed.

Case 5, CAB: A is next to B so this is allowed.

Case 6, CBA: A is next to B so this is allowed.

Therefore there are four possible seating arrangements.

2 A prime number is an integer which has two and only two distinct factors. 1 is not a prime number.

$n = \{20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30\}$

Case 1, $n = 20$: the factors of 20 are 1, 2, 4, 5, 10, 20 so 20 is not prime.

Case 2, $n = 21$: the factors of 21 are 1, 3, 7, 21 so 21 is not prime.

Case 3, $n = 22$: the factors of 22 are 1, 2, 11, 22 so 22 is not prime.

Case 4, $n = 23$: the factors of 23 are 1, 23 so 23 is prime.

Case 5, $n = 24$: the factors of 24 are 1, 2, 3, 4, 6, 8, 12, 24 so 24 is not prime.

Case 6, $n = 25$: the factors of 25 are 1, 5, 25 so 25 is not prime.

Case 7, $n = 26$: the factors of 26 are 1, 2, 13, 26 so 26 is not prime.

Case 8, $n = 27$: the factors of 27 are 1, 3, 9, 27 so 27 is not prime.

Case 9, $n = 28$: the factors of 24 are 1, 2, 4, 7, 14, 28 so 28 is not prime.

Case 10, $n = 29$: the factors of 24 are 1, 29 so 29 is prime.

Case 11, $n = 30$: the factors of 24 are 1, 2, 3, 5, 6, 10, 15 so 30 is not prime.

Consequently there are two prime numbers, 23 and 29, between 20 and 30 inclusive.

3 $n = \{1, 4, 9, 16, 25, 36, 49, 64, 81, 100\}$

Consequently there are no square numbers below 100 ending in 7.

4 $n = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

Case 1, $n = 1$: $(1)^2 + 3(1) + 19 = 23$, which is prime.

Case 2, $n = 2$: $(2)^2 + 3(2) + 19 = 29$, which is prime.

Case 3, $n = 3$: $(3)^2 + 3(3) + 19 = 37$, which is prime.

Case 4, $n = 4$: $(4)^2 + 3(4) + 19 = 47$, which is prime.

Case 5, $n = 5$: $(5)^2 + 3(5) + 19 = 59$, which is prime.

Case 6, $n = 6$: $(6)^2 + 3(6) + 19 = 73$, which is prime.

Case 7, $n = 7$: $(7)^2 + 3(7) + 19 = 89$, which is prime.

Case 8, $n = 8$: $(8)^2 + 3(8) + 19 = 107$, which is prime.

Case 9, $n = 9$: $(9)^2 + 3(9) + 19 = 127$, which is prime.

Consequently, the expression $n^2 + 3n + 19$ is prime for all positive integer values of n below 10.

5 $x = \{1, 3, 5\}$ and $y = \{1, 3, 5\}$

Case 1: $x + y = 1 + 1 = 2 = 2 \times 1$

Case 2: $x + y = 1 + 3 = 4 = 2 \times 2$

Case 3: $x + y = 1 + 5 = 6 = 2 \times 3$

Case 4: $x + y = 3 + 1 = 4 = 2 \times 2$

Case 5: $x + y = 3 + 3 = 6 = 2 \times 3$

Case 6: $x + y = 3 + 5 = 8 = 2 \times 4$

Case 7: $x + y = 5 + 1 = 6 = 2 \times 3$

Case 8: $x + y = 5 + 3 = 8 = 2 \times 4$

Case 9: $x + y = 5 + 5 = 10 = 2 \times 5$

So if x and y are odd positive integers less than 7 then their sum is always divisible by 2.

6 Case 1, 1, 1 and 9: $1 + 1 < 9$, so two sides have a sum shorter than the third.

Case 2, 1, 2 and 8: $1 + 2 < 8$, so two sides have a sum shorter than the third.

Case 3, 1, 3 and 7: $1 + 3 < 7$, so two sides have a sum shorter than the third.

Case 4, 1, 4 and 6: $1 + 4 < 6$, so two sides have a sum shorter than the third.

Case 5, 1, 5 and 5: $1 + 5 > 5$ and $5 + 5 > 1$, so any two sides sum to more than the third.

Case 6, 2, 2 and 7: $2 + 2 < 7$, so two sides have a sum shorter than the third.

Case 7, 2, 3 and 6: $2 + 3 < 6$, so two sides have a sum shorter than the third.

Case 8, 2, 4 and 5: $2 + 4 > 5$, $2 + 5 > 4$ and $4 + 5 > 2$, so any two sides sum to more than the third.

There are two distinct triangles: 1, 5, 5 and 2, 4, 5.

7 Case 1, 1, 1, 1 and 7: $1 + 1 + 1 < 7$, so three sides have a sum shorter than the fourth.

Case 2, 1, 1, 2 and 6: $1 + 1 + 2 < 6$, so three sides have a sum shorter than the fourth.

Case 3, 1, 1, 3 and 5: $1 + 1 + 3 = 5$, so three sides have a sum equal to the fourth.

Case 4, 1, 1, 4 and 4: $1 + 1 + 4 > 4$ and $1 + 4 + 4 > 1$, so any three sides have a sum greater than the fourth.

Case 5, 1, 2, 2 and 5: $1 + 2 + 2 = 5$, so three sides have a sum equal to the fourth.

Case 6, 1, 2, 3 and 4: $1 + 2 + 3 > 4$, $1 + 2 + 4 > 3$, $1 + 3 + 4 > 2$ and $2 + 3 + 4 > 1$, so any three sides have a sum greater than the fourth.

Case 7, 1, 3, 3 and 3: $1 + 3 + 3 > 3$ and $3 + 3 + 3 > 1$, so any three sides have a sum greater than the fourth.

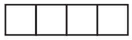
Case 8, 2, 2, 2 and 4: $2 + 2 + 2 > 4$ and $2 + 2 + 4 > 2$, so any three sides have a sum greater than the fourth.

Case 9, 2, 2, 3 and 3: $2 + 2 + 3 > 3$ and $2 + 3 + 3 > 2$, so any three sides have a sum greater than the fourth.

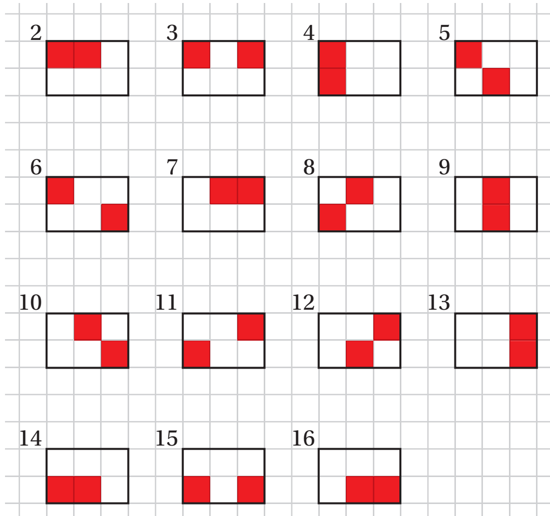
Consequently, there are five quadrilaterals with sides of integer length that have a perimeter of 10.

8 Note that any tetronimo must fit into a 1 by 4 rectangle or a 2 by 3 rectangle (neither 2 by 5 or 3 by 3 would be possible).

Case 1: There is only one possibility in a 1 by 4 rectangle: four in a row.



For cases 2 to 16, consider 2 by 3 rectangles with two squares removed.



Case 2: L shape

Case 3: T shape

Case 4: 2 by 2 square

Case 5: Squares are not all joined by edges

Case 6: S shape

Case 7: L shape

Case 8: Squares are not all joined by edges

Case 9: Squares are not all joined by edges

Case 10: Squares are not all joined by edges

Case 11: S shape

Case 12: Squares are not all joined by edges

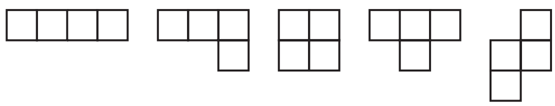
Case 13: Square

Case 14: L shape

Case 15: T shape

Case 16: L shape

Consequently, there are five possible tetronimos.



9 The first team could have scored 0, 1 or 2 before half-time.

The second team could have scored 0 or 1 before half-time.

The half-time score could be:

0–0, 0–1, 1–0, 1–1, 2–0, 2–1

Consequently there are six possible half-time scores for a match which finishes 2–1.

10 $n = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

Case 1: $n^6 - n = 1 - 1 = 0 = 2 \times 0$

Case 2: $n^6 - n = 64 - 2 = 62 = 2 \times 31$

Case 3: $n^6 - n = 729 - 3 = 726 = 2 \times 363$

Case 4: $n^6 - n = 4096 - 4 = 4092 = 2 \times 2046$

Case 5: $n^6 - n = 15\,625 - 5 = 15\,620 = 2 \times 7810$

Case 6: $n^6 - n = 46\,656 - 6 = 46\,650 = 2 \times 23\,325$

Case 7: $n^6 - n = 117\,649 - 7 = 117\,642 = 2 \times 58\,821$

Case 8: $n^6 - n = 262\,144 - 8 = 262\,136 = 2 \times 131\,068$

Case 9: $n^6 - n = 531\,441 - 9 = 531\,432 = 2 \times 265\,716$

So if n is a positive integer less than 10 then $n^6 - n$ is a multiple of 2.

Exercise 11.3A — page 296

1 **a** A rectangle with sides of 10 cm and 26 cm is not a square.

b –26 is not positive.

c When –26 is doubled, the answer is smaller (–52).

d –10 cubed is not positive.

e 2 is a prime number but is not odd.

f $\log(1 + 1) = \log 2$ but $\log 1 + \log 1 = 0$

g $(1 + 1)^2 = 4$ but $1^2 + 1^2 = 2$

2 $2^3 < 3^2$. 3 is the only counter example.

3 When $x = -4$, $x^2 + 8x + 15 = (-4)^2 + 8(-4) + 15 = -1$

4 ABCED has Carlos not next to Dupika.

5 Any prime number multiplied by 2 will be even (e.g. $2 \times 5 = 10$).

6 If $x + y = 1 + 3 = 4$, which is even, x and y don't have to be even.

7 2 is a prime number that is not odd.

8 If $x = \sqrt{2}$ and $y = 2\sqrt{2}$ then $xy = \sqrt{2} \times 2\sqrt{2} = 4$ which is rational.

9 If $x = -3$ and $y = 3$ then $x^2 = y^2$ but $x \neq y$.

Exam-style questions 11 — page 298

1 $(3n + 5)^2 - (3n - 5)^2 = 9n^2 + 30n + 25 - 9n^2 + 30n - 25$

$= 60n$

which is a multiple of 12.

2 The 12 ways are: SSUM, SSMU, SUSM, SMSU, SUMS, SMUS, USSM, MSSU, USMS, MSUS, UMSS and MUSS.

3 $x = \{2, 4, 6\}$ and $y = \{2, 4, 6\}$

Case 1: $x - y = 2 - 2 = 0$

Case 2: $x - y = 2 - 4 = -2$

Case 3: $x - y = 2 - 6 = -4$

Case 4: $x - y = 4 - 2 = 2$

Case 5: $x - y = 4 - 4 = 0$

Case 6: $x - y = 4 - 6 = -2$

Case 7: $x - y = 6 - 2 = 4$

Case 8: $x - y = 6 - 4 = 2$

Case 9: $x - y = 6 - 6 = 0$

All of these differences are divisible by 2.

4 When n is odd, the integers can be written as $a - \left(\frac{n-1}{2}\right), a - \left(\frac{n-1}{2} - 1\right), \ldots a - 2, a - 1, a, a + 1, a + 2, \ldots, a + \left(\frac{n-1}{2} - 1\right), a + \left(\frac{n-1}{2}\right)$, where a is the middle value (or the mean of the first and last terms) and is an integer. These have a sum of na, which is a multiple of n.

When n is even, the integers can be written as $a - \left(\frac{n-1}{2}\right), a - \left(\frac{n-1}{2} - 1\right), \ldots a - 2, a - 1, a, a + 1, a + 2, \ldots, a + \left(\frac{n-1}{2} - 1\right), a + \left(\frac{n-1}{2}\right)$, where a is the middle value (or the mean of the first and last terms) but is not an integer. These also have a sum of na, but this is not a multiple of n, because a is not an integer.

5 The three-digit cube numbers are:

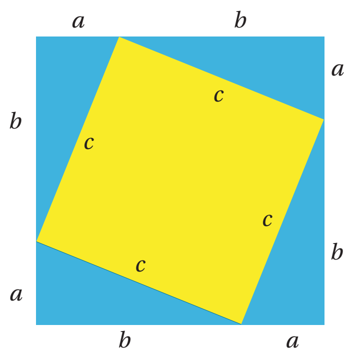
$5^3 = 125 \quad 6^3 = 216 \quad 7^3 = 343 \quad 8^3 = 512 \quad 9^3 = 729$

343 is the only palindromic three-digit cube number.

6 When $\theta = \frac{\pi}{2}$, $\sin 2\left(\frac{\pi}{2}\right) = 0$ whereas $2 \sin\left(\frac{\pi}{2}\right) = 2$.

7 When $n = 5$, $n^2 - n + 1 = 5^2 - 5 + 1 = 21$ which is not prime.

8



Area of large square – area of four triangles = area of small square

$$(a + b)^2 - 2ab = c^2$$
$$a^2 + 2ab + b^2 - 2ab = c^2$$
$$a^2 + b^2 = c^2$$

9 If $x = -4$ and $y = 3$, then $x < y$ but $x^2 > y^2$.

12 Data presentation and interpretation

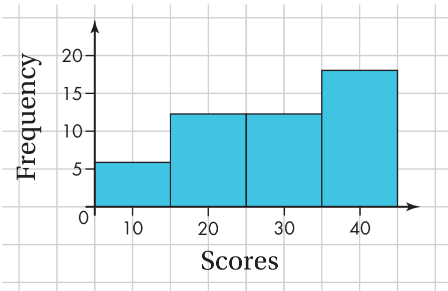
Prior knowledge page 300

1 a The data is quantitative as it is numerical, and discrete as there are distinct individual values.

b Mean = 32.88, median = 35, mode = 41 and range = 35

c

Grouped scores	Frequency
10–	6
20–	13
30–	13
40–	18



The results show a higher frequency or modal group 40–49.

Using the large data set 12.1 page 303

a Heathrow

	1987	2015
Mean	11.63	13.16
Median	11.1	13.1

2015 was a hotter year on average compared with 1987.

b Leuchars

	1987	2015
Mean	9.28	9.49
Median	9.0	9.6

Student's own comments about trend and whether longitude/latitude affects the temperatures.

c Beijing

May	1987	2015
Mean	19.87	21.57
Median	20.8	22.8

This data shows that there has been an increase in temperature from 1987 to 2015, which was also observed in the UK.

Exercise 12.1A page 306

1 a Daily rainfall in Lincoln is continuous, as you measure it.

b Monthly texts you send on your mobile is discrete, as you can count them.

c The number of burgers sold in a fast-food restaurant is discrete, as you can count them.

d The duration of a marathon is continuous, as you time it.

e The ages of the teachers in your school is continuous, as age is a continuous measure.

2 a Gender is qualitative, as a category.

b Height is quantitative, as has a numerical meaning.

c GCSE grades in maths is qualitative, as a category.

d Examination score in maths is quantitative, as it has a numerical meaning.

e Waist size is quantitative, as it has a numerical meaning.

f Car owner is qualitative, as a category.

g Weekly self-study time is quantitative, as it has a numerical meaning.

3 Various ideas such as height, weight, texts sent, music tracks owned, etc.

4 a 9 is the mode, as most common.

b 9 is the median when the data is put in numerical order.

c 13.05 when the data is added and divided by 20.

d The mean takes into account all the visits made so it is the best measure to use.

5 a Fiona's results sum to 22.4 mm. 22.4 + 84.6 = 107. $\frac{107}{28} = 3.8$

b No effect, as the total remains the same.

6

Number of times stopped	Number of journeys	fx
$\leqslant 1$	3	3
2	5	10
3	11	33
4	21	84
5	22	110
6	17	102
7	14	98
8	7	56
	100	496

Median = $100 + \frac{1}{2} = 50.5$th value; this is 5

Mode = 5 as the most frequent

Mean = $\frac{496}{100} = 4.96$

The best average to represent the data is the mean, as it uses all the data.

7

Length, x (cm)	Frequency	Midpoint	fx
$7.5 \leqslant x < 10$	30	8.75	262.5
$10 \leqslant x < 15$	70	12.5	875
$15 \leqslant x < 20$	100	17.5	1750
$20 \leqslant x < 30$	80	25	2000
$30 \leqslant x < 35$	40	32.5	1300
	320		6187.5

Mean = $\frac{6187.5}{320} = 19.3$ cm

8

Height (cm)	Frequency	fx
$100 < h \leqslant 120$	5	550
$120 < h \leqslant 140$	4	520
$140 < h \leqslant 160$	12	1800
$160 < h \leqslant 180$	13	2210
$180 < h \leqslant 200$	8	1520
Total	42	6600

Estimate of the mean = $\frac{6600}{42} = 157.14$

Exercise 12.2A page 311

1 **a** Mean = 8, standard deviation = 4

b Mean = 70, standard deviation = 14.14

c Mean = 13, standard deviation = 3.63

2 **a** Mean = 5, standard deviation = 2.19

b Mean = 16.5, standard deviation = 3.07

3 **a** Mean = 6.06, standard deviation = 1.12

b Both median and mode = 6. All three averages are similar as symmetrical distribution.

4 $\Sigma x = 27$, $\Sigma x^2 = 245$, $n = 3$

Mean = $\Sigma x = \frac{27}{3} = 9$

Standard deviation = $\sqrt{\frac{\Sigma x^2 - n\bar{x}^2}{n}} = \sqrt{\frac{245 - 3.81}{3}}$

$= 0.82$

5 Mean = $\frac{102}{15} = 6.8$

Variance = $\frac{1181 - 693.6}{15} = 32.5$

6 Mean = $\frac{12}{20} = 0.6$

Standard deviation = $\sqrt{\frac{144 - 7.2}{20}} = 2.62$

7 **a** Town route: $\Sigma x = 100$, $\Sigma x^2 = 2106$, $n = 5$

Mean = $\Sigma x = \frac{100}{5} = 20$

Standard deviation = $\sqrt{\frac{2106 - 2000}{5}} = 4.60$

Country route: $\Sigma x = 100$, $\Sigma x^2 = 2010$, $n = 5$

Mean = $\Sigma x = \frac{100}{5} = 20$

Standard deviation = $\sqrt{\frac{2010 - 2000}{5}} = 1.41$

b The country route is better as the average times are the same but the country route is more consistent.

8 **a** $\Sigma x = 16.6$, $\Sigma x^2 = 34.72$, $n = 8$

Mean = $\frac{16.6}{8} = 2.075$

Standard deviation = $\sqrt{\frac{34.72 - 34.445}{5}} = 0.185$

b No ball bearing is 2 standard deviations either side of the mean, so consistent results.

Using the large data set 12.2 page 313

Student's own justification. Suggestions are:

- First, is to remove all entries marked as 'tr' as this will not be recognised.
- Then calculate the outliers by finding the mean and then adding or subtracting 3 standard deviations.

Mean = 2.439 072 848

Standard deviation = 5.059 775 667

Mean + 3SD = 17.618 399 85

Mean – 3SD = –12.740 254

Remove the data entries above 17.6 mm.

Remaining data is clean.

Using the large data set 12.3 page 314

a 1987: mean = 92.74, SD = 5.65
2015: mean = 93.06, SD = 5.45

b Check for omissions.
Check for outliers using mean ± 3SD and omit these entries.
The SD could be reduced if outliers are omitted as data will be much more closely bunched, and the mean would be a much more accurate reflection of the data set.

c Mean 17.6
Standard deviation 2.58
Mean + 3SD 25.381
Mean – 3SD 8.899
No need to clean the data as all the points are within 3SD of the mean.

Using the large data set 12.4 page 317

Students own code. Suggestions: $y = \left(\frac{x}{100}\right) - 10$ or $y = \frac{(x - 1010)}{2}$

Mean: 1016.42 hPa

Standard deviation: 3.994 hPa

Exercise 12.2B page 317

1 Mean of x:

$60.2 = 1.4x - 15$

$\bar{x}\ 60.2 + \frac{15}{1.4} = 53.71$

Standard deviation:

$4.5 = 1.4x$

$x = \frac{4.5}{1.4} = 3.21$

2 **a** Mean $= \frac{418.3}{97} = 4.31$

Standard deviation = 2.29

b Mean = 3.831

Standard deviation = 7.1

3 **a** Mean = 67.07

Standard deviation = 9.97

b Mean = 19.48

Standard deviation = 5.54

4 Mean = 9.03 × 1.15 = 10.38 mph

Standard deviation = 3.92 × 1.15 = 4.51 mph

Exercise 12.3A page 323

1

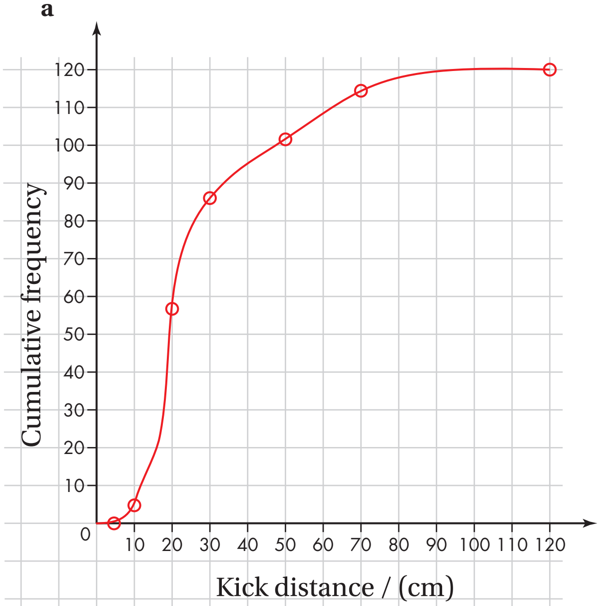
Number in family	Number of families	CF
1	15	15
2	20	35
3	22	57
4	23	80
5	11	91
6	4	95

Median = 48th value; this lies in the '3 in family' group.

2

Kick distance, l (m)	Upper bound	CF
$5 \leqslant l < 10$	10	5
$10 \leqslant l < 20$	20	58
$20 \leqslant l < 30$	30	87
$30 \leqslant l < 50$	50	102
$50 \leqslant l < 70$	70	113
$70 \leqslant l < 120$	120	120

a



b $40 - 5 = 35$

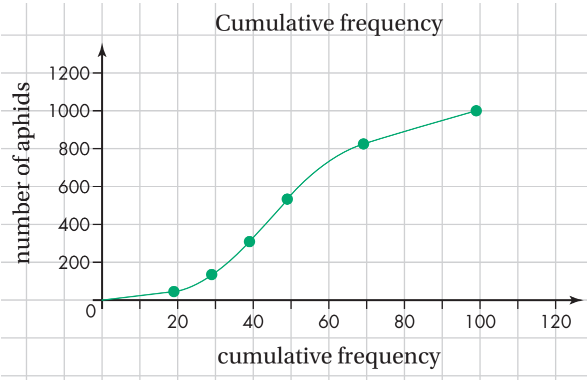
Lies in $10 \leqslant l < 20$ so $10 + \frac{35}{53} \times 10 = 16.6$ m

c

Kick distance, l (m)	Midpoint	Frequency	FX
$5 \leqslant l < 10$	7.5	5	37.5
$10 \leqslant l < 20$	15	53	795
$20 \leqslant l < 30$	25	29	725
$30 \leqslant l < 50$	40	15	600
$50 \leqslant l < 70$	60	11	660
$70 \leqslant l < 120$	95	7	665

$\text{Mean} = \frac{3482.5}{120} = 29.02$

3 a



b Median = 500.5th value

Lies in 40–49

$39 + \frac{192.5}{225} \times 10 = 47.56$

c $\text{Mean} = \frac{50545}{1000} = 50.55$

4 a Jenson: Q_0 3, Q_1 12, Q_2 31, Q_3 55, Q_4 66

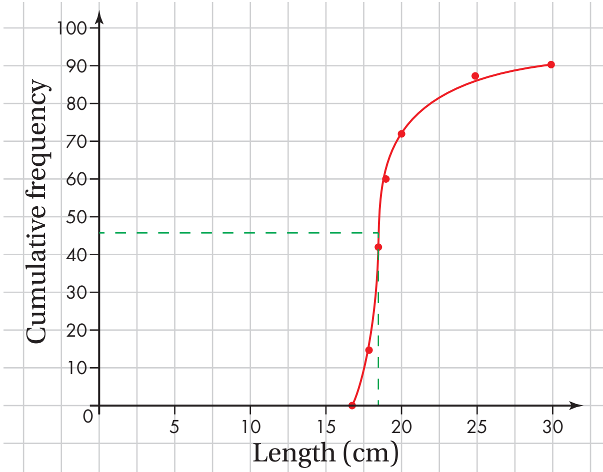
Molly: Q_0 12, Q_1 17, Q_2 34, Q_3 57, Q_4 98

b Medians are similar but Molly's is slightly higher, suggesting she scores more runs. Larger range for Molly, suggesting Jenson is more consistent. Jenson has a smaller IQR suggesting again that Jenson is more consistent.

5 a

Length (cm)	Frequency	Upper class	CF
17.5–17.9	15	17.95	15
18.0–18.4	27	18.45	42
18.5–18.9	18	18.95	60
19.0–19.9	12	19.95	72
20.0–24.9	15	24.95	87
25.0–29.9	4	29.95	91

b



c Using the graph, estimated median is between 17 and 19 cm.

d Median = 46th value

Within 18.5–18.9

$18.45 + \frac{4}{18} \times 0.5 = 18.56$

e Comparison should mention the fact that they are similar but interpolation is more accurate.

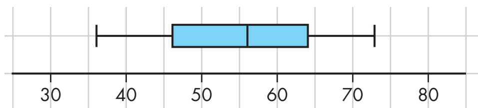
f Mean = 19.49 which is higher than the median as it gets affected by the higher values at the upper end of the data.

6 a Mode = 60

b Q_0 36, Q_1 46, Q_2 56, Q_3 64, Q_4 73

IQR = 18

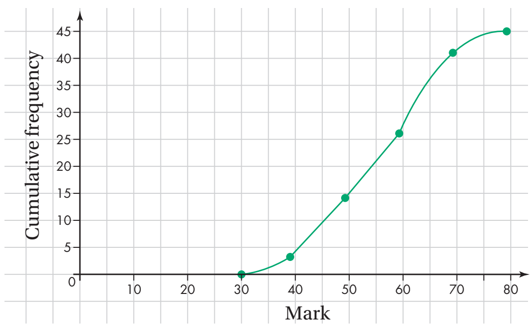
c



d $Q_3 + 1.5 \times \text{IQR} = 64 + 27 = 91$, no outliers

$Q_1 - 1.5 \times \text{IQR} = 46 - 27 = 19$, no outliers

e



f 28; This measure is useful because it omits extreme values.

7 **a** A: True, both medians are 34.

B: True, the Q_1 values have a difference of 5.

C: False, the difference between the IQRs is $\frac{5}{6}$.

D: False, the values are 7 and 9.

b Class A is more consistent as it has a smaller range so less extremes. Class B has a smaller IQR so the middle 50% of the data is more consistent.

8 **a** **i** 63 minutes

ii 120 minutes

iii $80 - 46 = 34$

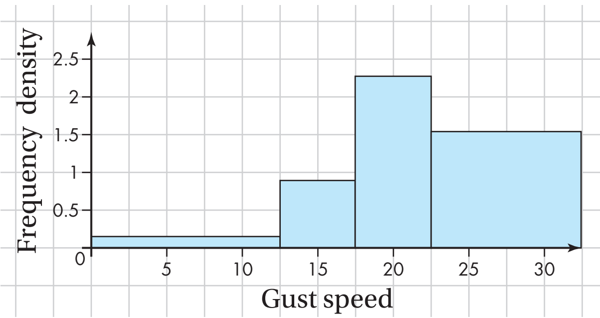
iv Assuming equally distributed 42.

b Cannot tell with boxplot (why?) but using the graph, 25.

Using the large data set 12.5 page 327

a

Gust	Frequency	FD
0–12	4	0.08
13–17	9	0.8
18–22	8	2.2
23–33	7	1.5



b Student's own value from their table, for example $\frac{32}{2} = 16\text{th value} = 22.5$

c The daily maximum gust is a spontaneous reading and therefore Camborne cannot be used to make predictions as position in UK will have different maximum gusts during the day.

d The daily maximum gust is a spontaneous reading and therefore Camborne cannot be used to make predictions as Jacksonville is in the USA and will have different maximum gusts during the day.

Exercise 12.3B page 327

1 **a** 3 and 25.5

b Width $= 9\frac{38}{5}$ and height $= \frac{25}{133}$

c Estimate of the mean = 15.05

d 40 students so the 20th value; lies in 11–15

$$\frac{\text{Median} - 10.5}{20 - 10} = \frac{15.5 - 10.5}{21 - 10}$$

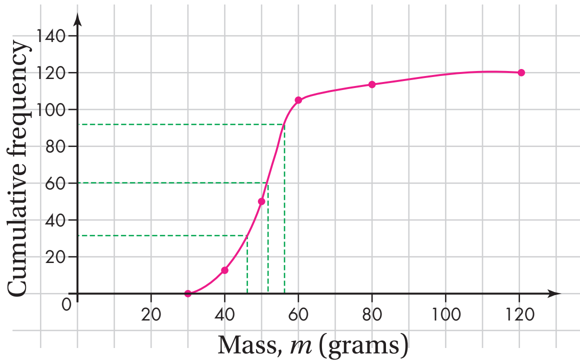
$$\frac{\text{Median} - 10.5}{10} = \frac{5}{11}$$

$$\text{Median} = 10\left(\frac{5}{11}\right) + 10.5$$

$$10.5 + \frac{5}{11} \times 10 = 15.04$$

2 **a**

Mass, m (grams)	Frequency	Upper bound	CF
$30 \leqslant m < 40$	13	40	13
$40 \leqslant m < 50$	37	50	50
$50 \leqslant m < 60$	56	60	106
$60 \leqslant m < 80$	8	80	114
$80 \leqslant m < 120$	6	120	120

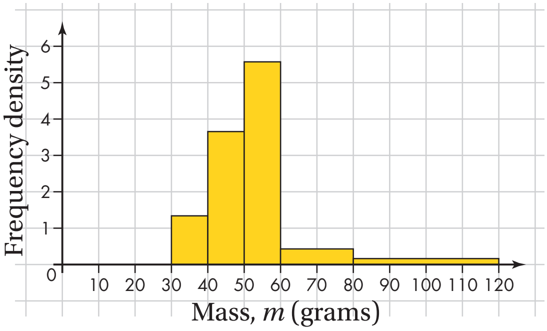


i The median mass of a cookie is about 51 g.

ii The lower and upper quartiles masses of the cookies are about 45 g and 59 g.

b

Mass, m (grams)	FD
$30 \leqslant m < 40$	1.3
$40 \leqslant m < 50$	3.7
$50 \leqslant m < 60$	5.6
$60 \leqslant m < 80$	0.4
$80 \leqslant m < 120$	0.15



c $Q_3 + 1.5 \times \text{IQR} = 59 + 1.5(14) = 80\,\text{g}$

3 **a** Width = 1.6 cm

b Height 2.3 cm

4 **a** Continuous data

b

Distance (km)	Number of people	FD
40–49	67	$\frac{67}{10} = 6.7$
50–59	124	$\frac{124}{10} = 12.4$
60–64	4023	$\frac{4023}{5} = 804.6$
64–69	2981	$\frac{2981}{5} = 596.2$
70–84	89	$\frac{89}{14} = 5.9$
85–149	75	$\frac{75}{65} = 1.2$

5 **a** Width $= \frac{10}{5} \times 1.5 = 3\,\text{cm}$

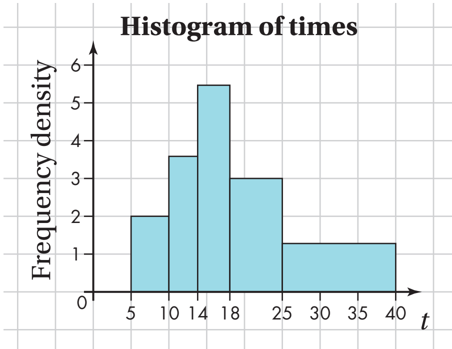
Height $= \frac{0.4}{5.8} \times 6 = 0.41\,\text{cm}$

b $5 + \frac{24}{29} \times 5 = 9.14\,\text{kg}$

c Mean = 10.7 kg

Standard deviation = 6.84 kg

6



a

t	Frequency	Frequency density
5–10	10	2
10–14	15	3.75
14–18	22	5.5
18–25	21	3
25–40	18	1.2

b $5 \times 3 + 18 = 33$ people

c Mean = 19.1 minutes

d Standard deviation = 8.1 minutes

e $Q_1 = 13.2$

$Q_2 = 17.3$

$Q_3 = 24$

7 **a** Width $= \frac{70}{40} \times 2 = 3.5\,\text{cm}$

Height $= \frac{24}{20} \times \frac{(2 \times 7)}{3.5} = 4.8\,\text{cm}$

b Median $= 240 + \frac{24}{30} \times 40 = £272$

c Mean = £306.6

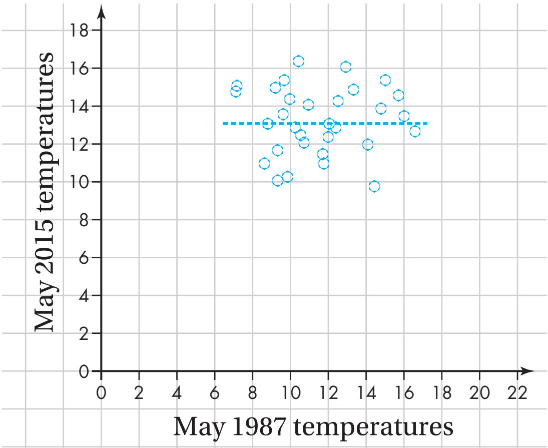
Standard deviation = £125.9

Using the large data set 12.6 page 332

a Standard deviation = 2.522 °C

b Less than 4.07 or greater than 19.20

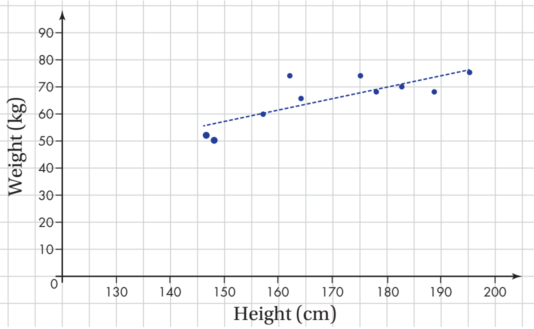
c



d From the data in May at Heathrow there is no evidence to suggest global warming as the daily mean temperatures are very much randomly distributed with no correlation.

Exercise 12.3C page 332

1 **a, c**



b Positive correlation: as height increases, so does weight.

d 78 ± 1 kg

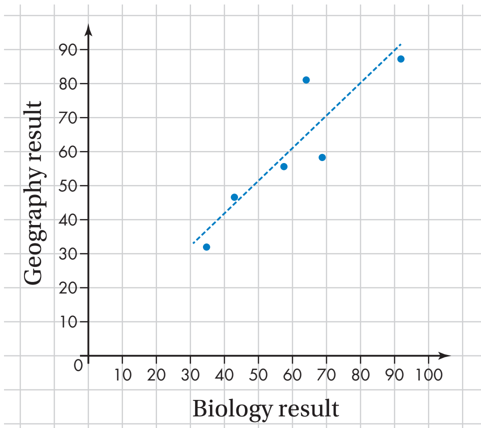
2 **a** An example could be 'Weight and strength'

b An example could be 'Alcohol consumption and judgment'

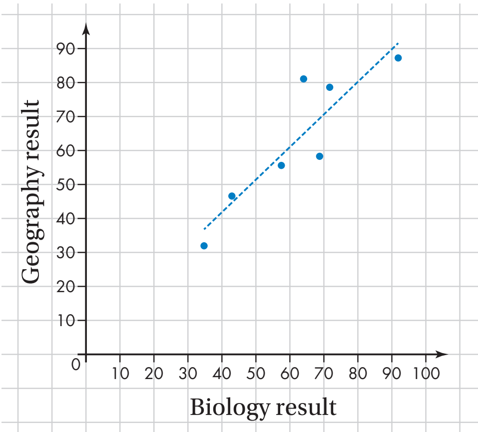
c An example could be 'Tests scores in French and geography'

d An example could be 'Amount of money spent on food and death through getting stuck in bedding'

3

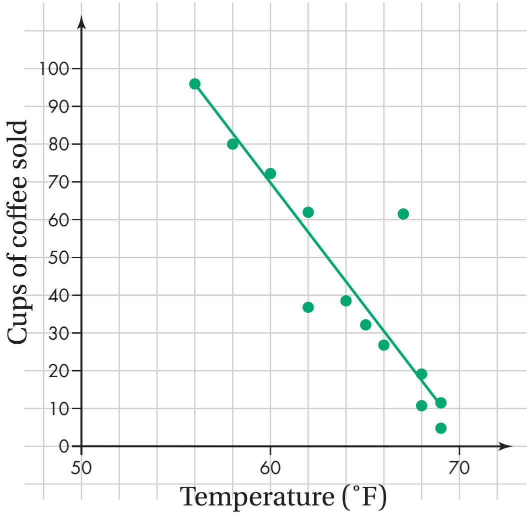


4 **a**



b The scatter graph doesn't indicate any outliers

5 **a** and **c**

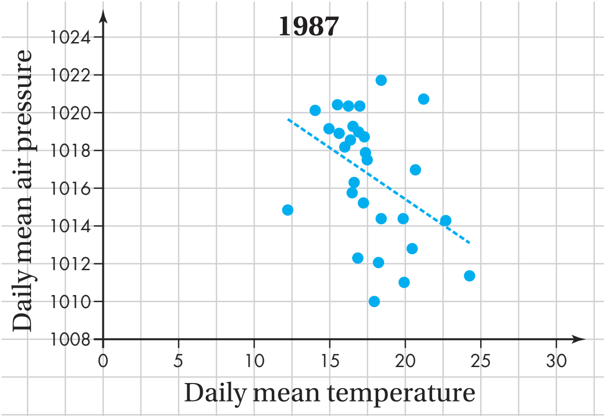


b Negative correlation: as temperature increases, coffees sold decreases

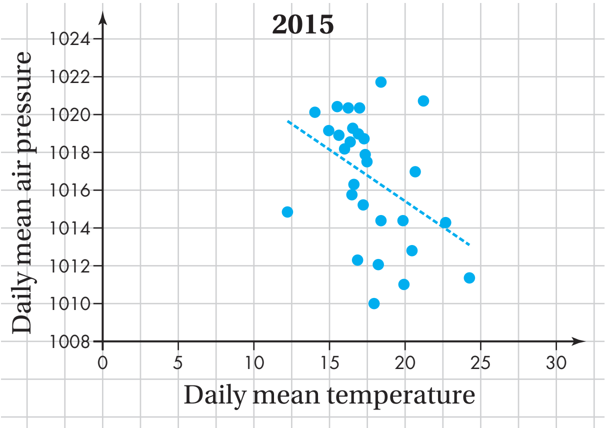
d From scatter diagram, 82 cups

Using the large data set 12.7 page 336

a



This suggests as temperature increases, air pressure decreases. There are generally 1026.2hPa air pressure every day but as temperature increases, the air pressure decreases by 0.54.



This suggests as temperature increases, air pressure decreases. There are generally 1027.9hPa air pressure every day but as temperature increases, the air pressure decreases by 0.49.

b 1987: $y = -0.5375x + 1026.2$
The pressure is expected to drop 0.5375hPa every one degree rise in temperature.
2015: $y = -0.4888x + 1027.9$
The pressure is expected to drop 0.4888hPa every one degree rise in temperature.

c Compared to 1987, 2015 seems to have been a warmer and more consistent October. 1987 has bigger drops in air pressure.

d You would not be able to use this data to make comparisons with Beijing, as Beijing is much further north so climatically not the same.

Exercise 12.3D page 337

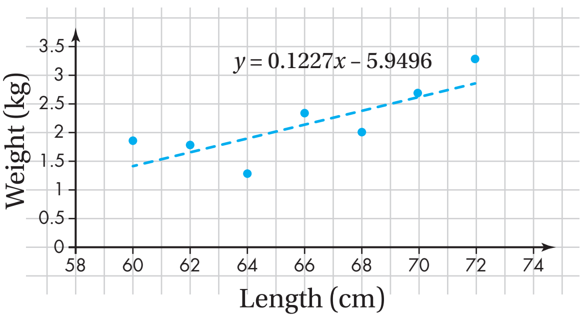
1 **a** a is the y intercept, b is the gradient

b 4034 g

c Extrapolation, hence unreliable as the data does not extend to 60 cm.

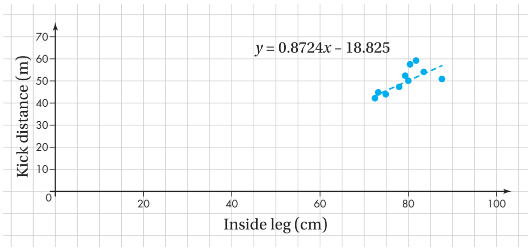
2 $y = 48.9 + 7.93x$
$y = 48.9 + 7.93(6.5)$
$= 100.4$ (extrapolation as the data does not extend this far, therefore not an accurate result)

3



$y = 0.12x - 5.95$
$y = 0.12(78) - 5.95$
$y = 3.41$ kg
Extrapolation, as beyond the measured data so not reliable.

4 **a**



b $y = 0.87x - 18.8$

c About 86 cm

d 42.1 m; this is extrapolation, so unreliable

5 **a** Positive correlation

b Student's response – about 90%

c Interpolation – inside the data set

d Humidity is generally 84.082% but as temperature rises humidity increases by 0.4% per °C.

e About 99.9%

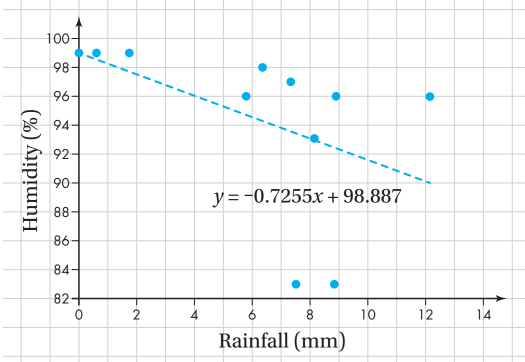
f Extrapolation – not in data set, cannot say for sure.

6 **a** IQR = 6.45

Low outliers = $1.8 - 1.5 \times 6.45 = -7.875$; this would be impossible so no low outliers.

High outliers = $8.25 + 1.5 \times 6.45 = 17.925$; no rainfall above this so no high outliers.

b



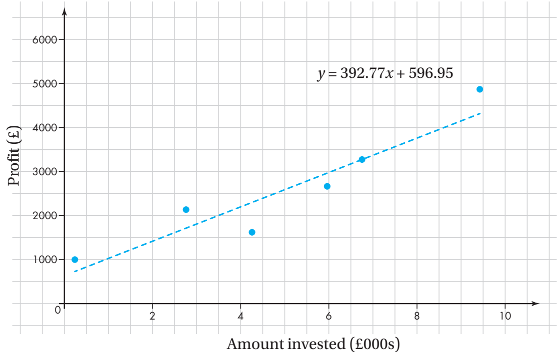
c Negative correlation: as rainfall increases, humidity decreases

d $h = 98.88 - 0.7255r$

e 92.1%

Exam-style questions 12 page 339

1 a Positive correlation



b $y = 597 + 393x$

For every £1000 invested, a return of £393 profit can be expected.

c £1971.65

2 $100 + 3(7) \geqslant 121$ cm or $100 - 3(7) \leqslant 79$ cm

3 a $\dfrac{4930}{17} = 290$ kg

$290 + 3(10.5) = 321.5$ kg

b New mean = 290.1 kg

4 a Q_1 121, Q_2 187, Q_3 260

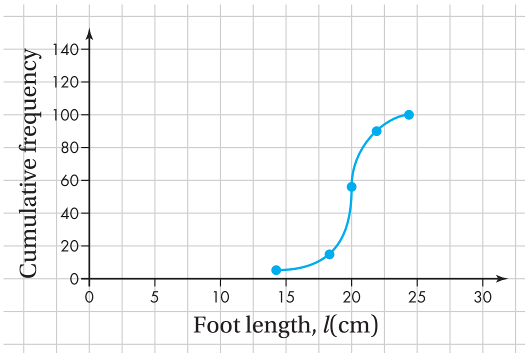
b $Q_3 + 1.5 \times \text{IQR} = 260 + 1.5(139) = 468.5$ and $121 - 1.5(139) = -87.5$ so no outliers

5 a $\dfrac{(100+1)}{2} = 50.5$th person

In the $19 \leqslant l < 21$ category

$19 + \dfrac{34}{38} \times 2 = 20.8$ cm

b



About 20 cm

c $\dfrac{20.8 - 20}{20.8} \times 100 = 3.8\%$

d Mean = 20.8

Standard deviation = 1.96

e Mean is best as no obvious outliers, and it uses all the available data.

6 a Width = 12.5 cm

b Height = 9.7 cm

c 52nd value: lies in 20–29 category so

$20 + \dfrac{32}{34} \times 10 = 29.4$ = median

26th value: lies in 20–29 category so

$20 + \dfrac{6}{34} \times 10 = 21.8$ = lower quartile

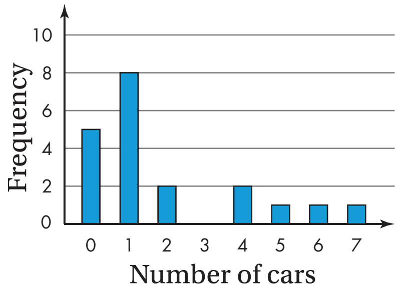
78th value: lies in 30–39 category so

$30 + \dfrac{24}{27} \times 10 = 38.9$ = upper quartile

IQR = 17.1

d Median is best due to the extreme values in the data set.

7



b Median is between the 10th and 11th value so is 1 car.

c Mean = $\dfrac{38}{20} = 1.9$

$S_{xx} = \sum fx^2 - n\bar{x}^2$

$S_{xx} = 158 - 20 \times 1.9^2 = 85.8$ $\sigma = \dfrac{\sqrt{85.8}}{20} = 2.07$

d $1.9 + 3(2.07) = 8.11$

The manufacturer would be concerned if a car is returned 9 or more times.

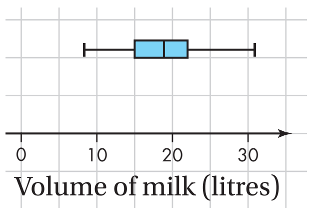
8 a Median = 19

Upper quartile = 22

Lower quartile = 15

Interquartile range = 7

Min value 8, maximum value 31



b Outlier = UQ + 1.5(IQR)

Outlier = 22 + 1.5(7) = 32.5, therefore no outliers

Outlier = LQ – 1.5(IQR)

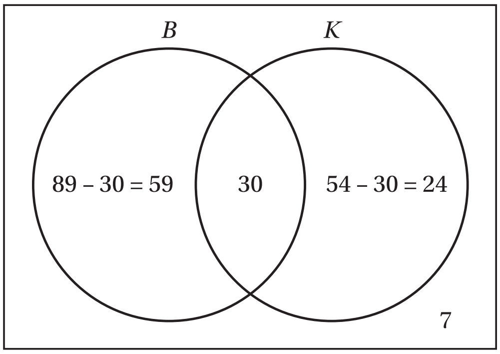
Outlier = 15 – 1.5(7) = 4.5, therefore no outliers

c Mean = 18.6L and median = 19L These values are consistent as there are no outliers in the data set.

13 Probability and statistical distributions

Prior knowledge **page 343**

1 a



b 7 members

2 a $\frac{1}{3}$

b $\frac{4}{15}$

c $\frac{3}{5}$

Exercise 13.1A **page 350**

1 a If there are 200 students in total, 46 must be in 'Brighton' house. The probability is therefore $\frac{46}{200} = 0.23$

b Probability of 'York' house = 0.31

Probability of 'Gretna' house = 0.26

Probability of not 'York' house or 'Gretna' house is therefore $1 - (0.31 + 0.26) = 0.43$

2 a Half of the tickets are even numbers so $\frac{1}{2}$ or 0.5

b Square ticket numbers are 1, 4, 9, 16, 25, 36, 49, 64, 81, 100. There are 10 square ticket numbers.

Probability of selecting a square ticket number $= \frac{10}{100} = 0.1$

Therefore the probability of not selecting a square ticket number $= 1 - 0.1 = 0.9$

c Valid ticket numbers are: 5, 15, 25, 35, 45, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 65, 75, 85, 95

There are 19 numbers in total.

Therefore the probability of selecting a number containing the digit 5 is $\frac{19}{100}$ or 0.19

3 $20 = 3x + 8$

$12 = 3x$

$x = 4$

Probability of selecting a black ball $= \frac{4}{20} = 0.2$

Therefore, the probability of not selecting a black ball $= 1 - 0.2 = 0.8$

4 a $P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.5 + 0.3 - 0.2 = 0.6$

b $P(A \cap B) = P(A) \times P(B)$

$0.2 = P(A) \times 0.3$

$P(A) = \frac{2}{3}$, so $P(A') = \frac{1}{3}$

Therefore, $P(A' \cap B) = P(A') \times P(B) = \frac{1}{3} \times 0.3 = 0.1$

5 a $P(S \text{ or } T) = 0.62$

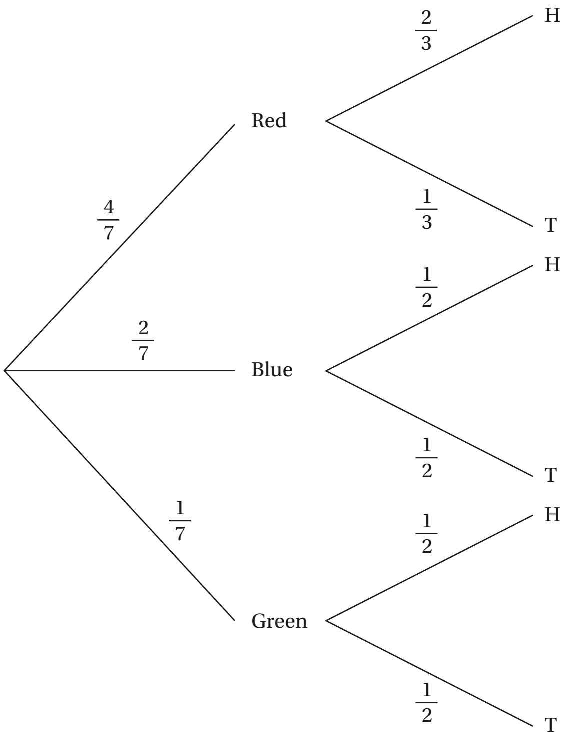
b $P(S' \text{ and } T') = 0.38$

6 a $P(A') = 1 - P(A) = 1 - 0.3 = 0.7$

b $P(B') = 1 - P(B) = 1 - 0.5 = 0.5$

c $P(A) + P(B') = 0.3 + 0.35 = 0.65$

7 a



7 b There are three combinations where a head can be obtained:

P(red and head), P(blue and head) and P(green and head)

$P(\text{red and head}) = \frac{4}{7} \times \frac{2}{3} = \frac{8}{21}$

$P(\text{blue and head}) = \frac{2}{7} \times \frac{1}{2} = \frac{1}{7}$

P(green and head) $= \frac{1}{7} \times \frac{1}{2} = \frac{1}{14}$

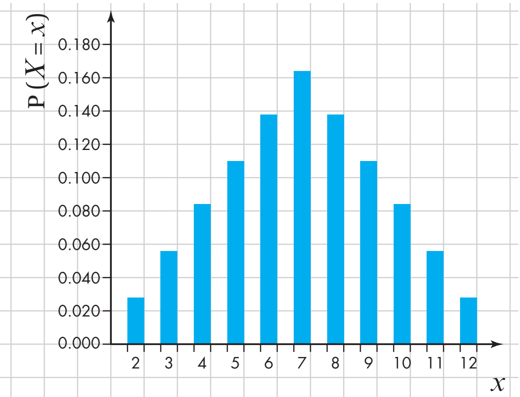
P(head) $= \frac{8}{21} + \frac{1}{7} + \frac{1}{14} = \frac{25}{42}$ or 0.595 (3 s.f.)

Exercise 13.2A page 354

1 a

x	$P(X=x)$	x	$P(X=x)$	x	$P(X=x)$
2	$\frac{1}{36}$	6	$\frac{5}{36}$	10	$\frac{1}{12}$
3	$\frac{1}{18}$	7	$\frac{1}{6}$	11	$\frac{1}{18}$
4	$\frac{1}{12}$	8	$\frac{5}{36}$	12	$\frac{1}{36}$
5	$\frac{1}{9}$	9	$\frac{1}{9}$		

b

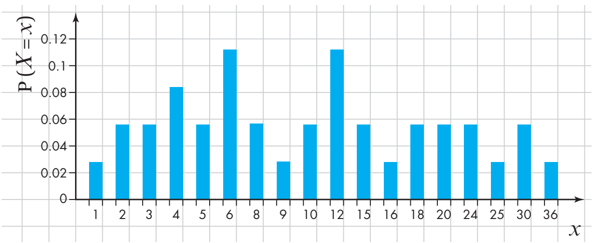


The distribution is unimodal and symmetrical.

2 a

a	$P(A=a)$	a	$P(A=a)$	a	$P(A=a)$
1	$\frac{1}{36}$	8	$\frac{1}{18}$	18	$\frac{1}{18}$
2	$\frac{1}{18}$	9	$\frac{1}{36}$	20	$\frac{1}{18}$
3	$\frac{1}{18}$	10	$\frac{1}{18}$	24	$\frac{1}{18}$
4	$\frac{1}{12}$	12	$\frac{1}{9}$	25	$\frac{1}{36}$
5	$\frac{1}{18}$	15	$\frac{1}{18}$	30	$\frac{1}{18}$
6	$\frac{1}{9}$	16	$\frac{1}{36}$	36	$\frac{1}{36}$

b

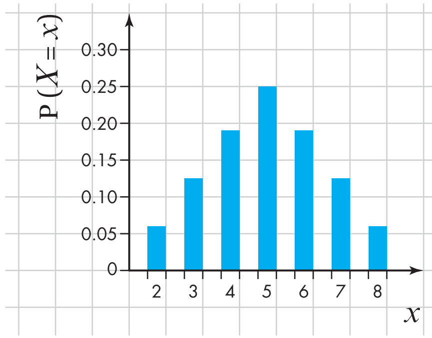


No distinguishable shape

3 a

x	2	3	4	5	6	7	8
$P(X=x)$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{3}{16}$	$\frac{1}{4}$	$\frac{3}{16}$	$\frac{1}{8}$	$\frac{1}{16}$

b



One mode and symmetrical

c Possible prime numbers are: 2, 3, 5, 7

Therefore P(X is a prime number)

$= \frac{1}{16} + \frac{1}{8} + \frac{1}{4} + \frac{1}{8} = \frac{9}{16}$ or 0.5625

4 a $1k + 2k + 3k + 4k = 1$

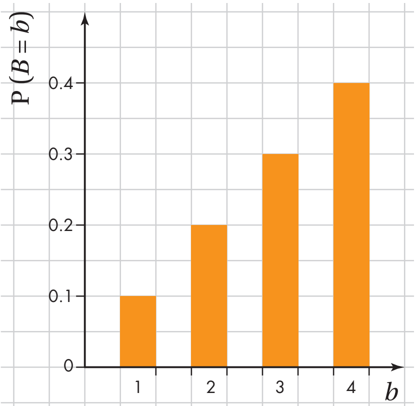
$10k = 1$

$k = 0.1$

b

b	1	2	3	4
$P(B=b)$	0.1	0.2	0.3	0.4

c



Negatively skewed distribution

5 a The probability for each value of x is equal over six possible outcomes.

Therefore $k = \frac{1}{6}$

b $P(X < 7) = P(X = 2) + P(X = 3) + P(X = 5)$

$= \frac{1}{6} + \frac{1}{6} + \frac{1}{6}$

$= \frac{1}{2}$ or 0.5

c $P(X \geqslant 3) = 1 - P(X < 3)$

$= 1 - P(X = 2)$

$= 1 - \frac{1}{6}$

$= \frac{5}{6}$

d $P(2 \leqslant X < 11) = P(X = 2) + P(X = 3) + P(X = 5) + P(X = 7)$

$= \frac{2}{3}$

6 a $P(X = x) = kx^2$

x	1	2	3	4	5
$P(X = x)$	$1k$	$4k$	$9k$	$16k$	$25k$
$P(X = x)$	$\frac{1}{55}$	$\frac{4}{55}$	$\frac{9}{55}$	$\frac{16}{55}$	$\frac{25}{55}$

Since $55k = 1$

$k = \frac{1}{55}$

b $P(X = x) = \frac{k}{x}$

x	1	2	3	4	5
$P(X = x)$	$\frac{k}{1}$	$\frac{k}{2}$	$\frac{k}{3}$	$\frac{k}{4}$	$\frac{k}{5}$
$P(X = x)$	0.438…	0.219…	0.146…	0.109…	0.088…

Since $\frac{60k}{60} + \frac{30k}{60} + \frac{20k}{60} + \frac{15k}{60} + \frac{12k}{60} = 1$

$\frac{137k}{60} = 1$

$137k = 60$

$k = \frac{60}{137}$

c $P(X = x) = \frac{x}{k}$

x	1	2	3	4	5
$P(X = x)$	$\frac{1}{k}$	$\frac{2}{k}$	$\frac{3}{k}$	$\frac{4}{k}$	$\frac{5}{k}$
$P(X = x)$	$\frac{1}{15}$	$\frac{2}{15}$	$\frac{3}{15}$	$\frac{4}{15}$	$\frac{5}{15}$

$\frac{15}{k} = 1$

$k = 15$

Exercise 13.2B — page 357

1 a

X	1	2	3	4	5
$P(X = x)$	0.15	0.2	0.5	0.1	0.05
$F(x)$	0.15	0.35	0.85	0.95	1

b

X	1	2	4	8
$P(X = x)$	0.25	0.25	0.25	0.25
$F(x)$	0.25	0.5	0.75	1

c

X	–3	–1	1	3
$P(X = x)$	0.8	0.1	0.02	0.08
$F(x)$	0.8	0.9	0.92	1

d

X	1	2	3	4	5
$P(X = x)$	0.2	$2a$	0.4	a	0.1
$F(x)$	0.2	$2a + 0.2$	$2a + 0.6$	$3a + 0.6$	$3a + 0.7$

2 a

X	1	2	3	4	5
$P(X = x)$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$
$F(x)$	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{4}{5}$	1

b $P(X < 3) = \frac{2}{5}$

c Using the cumulative distribution function table from **part a**:

$P(1 \leqslant X < 4) = P(X \leqslant 3) = \frac{3}{5}$ or 0.6

3 a

x	2	4	6	8
$F(x)$	0.3	0.6	0.8	1
$P(X = x)$	0.3	0.3	0.2	0.2

b

x	–5	0	5	10
$F(x)$	0.05	0.4	0.5	1
$P(X = x)$	0.05	0.35	0.1	0.5

c

x	2	5	6	22
$F(x)$	0.1	0.5	0.75	1
$P(X = x)$	0.1	0.4	0.25	0.25

4 Adding a row for $P(X = x)$ in terms of a:

x	1	2	3	4
$F(x)$	0.32	a	0.78	1
$P(X = x)$	0.32	0.23	0.23	0.22

Since $P(X = 2) = P(X = 3)$, $a - 0.32 = 0.78 - a$, $a = 0.55$

5 Adding a row for $P(R = r)$:

r	0	1	2
$F(r)$	$\frac{1}{8}$	$\frac{1}{2}$	1
$P(R = r)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{1}{2}$

6 a

s	1	2	4	6
$P(S = s)$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$

b Uniform probability distribution

c Unimodal, symmetrical

d P(both prime) $= \frac{1}{4} \times \frac{5}{6} = \frac{5}{24}$

7 a

x	1	2	3	4	5	6
$P(X=x)$	$\frac{1}{36}$	$\frac{3}{36}$	$\frac{5}{36}$	$\frac{7}{36}$	$\frac{9}{36}$	$\frac{11}{36}$

b $P(2 < X \leqslant 5) = P(X=3) + P(X=4) + P(X=5)$

$= \frac{5}{36} + \frac{7}{36} + \frac{9}{36}$

$= \frac{21}{36}$

8 a Substituting the values of x into the probability function,

$1225k = 1$

$k = \frac{1}{1225}$, as required

b Substituting $k = \frac{1}{1225}$ into the probability distribution table created in **part a**,

x	1	3	5	7	9
$P(X=x)$	$\frac{1}{1225}$	$\frac{27}{1225}$	$\frac{125}{1225}$	$\frac{343}{1225}$	$\frac{729}{1225}$

Therefore, $P(X \geqslant 7) = P(X=7) + P(X=9)$

$= \frac{343}{1225} + \frac{729}{1225}$

$= \frac{1072}{1225}$

9 Since the cumulative probabilities (F(y)) are calculated by adding the current and previous probabilities for each respective value of Y, the following simultaneous equations are derived:

$0.1 = a$ ①

$0.5 = a + b$ ②

$d = 0.3 + a + b$ ③

$1 = a + b + 0.3 + c$ ④

Substitute $a = 0.1$ into ②.

$0.5 = 0.1 + b$

$b = 0.4$

Substitute $a = 0.1$ and $b = 0.4$ into ③.

$d = 0.3 + 0.1 + 0.4$

$d = 0.8$

Substitute $a = 0.1$ and $b = 0.4$ into ④.

$1 = 0.1 + 0.4 + 0.3 + c$

$1 = 0.8 + c$

$c = 0.2$

Therefore $a = 0.1$, $b = 0.4$, $c = 0.2$ and $d = 0.8$

10 a Since all probabilities sum to 1,

$\frac{2}{18} + \frac{2k}{18} + \frac{7}{18} + \frac{k}{18} = 1$

Therefore $3k = 9$

$k = 3$, as required

b Substitute $k = 3$ in the previous probability distribution table,

x	2	3	4	6
$P(X=x)$	$\frac{2}{18}$	$\frac{6}{18}$	$\frac{7}{18}$	$\frac{3}{18}$
$F(x)$	$\frac{2}{18}$	$\frac{8}{18}$	$\frac{15}{18}$	1

Exercise 13.3A page 364

1 a i $P(X=2)$, $P(X=2) = {}^{10}C_2 \times 0.2^2 \times 0.8^8$

$= 0.302$ (3 d.p.)

ii $P(X=5)$, $P(X=5) = {}^{10}C_5 \times 0.2^5 \times 0.8^5$

$= 0.026$ (3 d.p.)

iii $P(X=10)$, $P(X=10) = {}^{10}C_{10} \times 0.2^{10} \times 0.8^0$

$= 0.000$ (3 d.p.)

b i $P(X=2)$, $P(X=2) = {}^{7}C_2 \times 0.8^2 \times 0.2^5$

$= 0.004$ (3 d.p.)

(actual answer is 0.004 300 8)

ii $P(X=5)$, $P(X=5) = {}^{7}C_2 \times 0.8^5 \times 0.2^2$

$= 0.275$ (3 d.p.)

iii $P(X<3)$, Since the binomial distribution is discrete, $P(X<3) = P(X=1) + P(X=2)$

$= 0.005$ (3 d.p.)

c i $P(X=18) = {}^{20}C_{18} \times 0.125^{18} \times 0.875^2$

$= 0.000$ (3 d.p.)

ii From cumulative binomial probabilities, $P(X \leqslant 5) = 0.969$ (3 d.p.)

iii $P(3 \leqslant X < 15)$, $P(3 \leqslant X < 15) = P(X \leqslant 14) - P(X \leqslant 2)$,

$= 0.9999\ldots - 0.5353\ldots$

$= 0.465$ (3 d.p.)

d i $P(X<4)$, Since the binomial distribution is discrete, $P(X<4) = P(X \leqslant 3) = 0.913$

ii $P(X=5)$, $P(X=5) = {}^{5}C_5 \times 0.4^5 \times 0.6^0$

$= 0.010$ (3 d.p.)

iii $P(0 \leqslant X < 3) = P(X<3)$

$= P(X \leqslant 2)$

$= 0.683$ (3 d.p.)

e **i** $P(X \leqslant 12) = 0.9999\ldots = 1.000$ (3 d.p.)

ii $P(5 \leqslant X < 8) = P(X \leqslant 7) - P(X \leqslant 4)$
$= 0.9973 - 0.8776$
$= 0.120$ (3 d.p.)

iii $P(2 \leqslant X < 13) = P(X \leqslant 12) - P(X \leqslant 1)$
$= 0.9999\ldots - 0.2187\ldots$
$= 0.781$ (3 d.p.)

2 Number of trials is 8, therefore $n = 8$

Probability of rolling a six is $\frac{1}{6}$, therefore $p = \frac{1}{6}$

Since there is a set number of independent trials with a consistent probability, the problem can be modelled using a binomial distribution.

Let X be the number of sixes rolled.

Therefore, $X \sim B(8, \frac{1}{6})$

$P(X = 3) = {}^{8}C_3 \times \left(\frac{1}{6}\right)^3 \times \left(\frac{5}{6}\right)^5 = 0.104$ (3 d.p.)

3 $P(X = 6) = {}^{10}C_6 \times (p)^6 \times (1 - p)^4$

4 **a** $P(X < 5) = P(X \leqslant 4)$
$= 0.842$ (3 s.f.)

b $P(X \geqslant 7) = 1 - P(X \leqslant 6)$
$= 1 - 0.9857$
$= 0.0143$ (3 s.f.)

5 **a** Number of students selected is 30, therefore $n = 30$

Probability of a student asking for paper is 3 out of 5, therefore $p = 0.6$

Since there is a set number of independent trials and a fixed probability of success, the problem can be modelled as a binomial distribution.

Let X be the number of students who asked for paper.

Therefore, $X \sim B(30, 0.6)$

$P(X = 10) = {}^{30}C_{10} \times (0.6)^{10} \times (0.4)^{20}$
$= 0.002$ (3 s.f.)

b $P(X < 20) = P(X \leqslant 19)$
$= 0.709$ (3 s.f.)

6 **a** Number of flights in the winter is 10, therefore $n = 10$

Probability of poor visibility is 25%, therefore $p = 0.25$

Since there is a set number of independent trials with a fixed probability of success, the problem can be modelled as a binomial distribution.

Let X be the number of poor visibility encounters.

Therefore, $X \sim B(10, 0.25)$

$P(X = 3) = {}^{10}C_3 \times (0.25)^3 \times (0.75)^7$
$= 0.250$ (3 s.f.)

b $P(X \geqslant 3) = 1 - P(X \leqslant 2)$
$= 1 - 0.5256$
$= 0.474$ (3 s.f.)

7 **a** Number of students is 20, therefore $n = 20$

Probability of having a birthday in January is $\frac{1}{12}$, therefore $p = \frac{1}{12}$

Since there is a set number of independent students and a fixed probability of having a birthday in January, the problem can be modelled as a binomial distribution.

Let X be the number of students with birthdays in January.

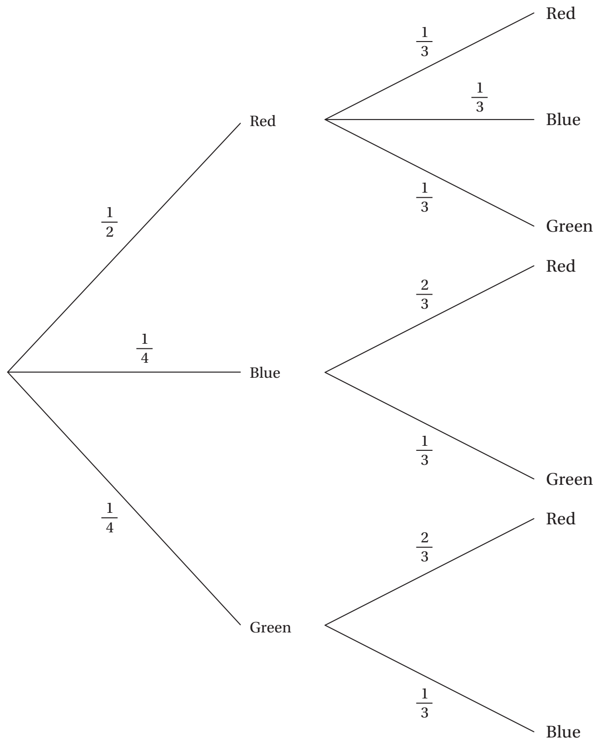
Therefore, $X \sim B(20, \frac{1}{12})$

$P(X = 1) = {}^{20}C_1 \times \left(\frac{1}{12}\right)^1 \times \left(\frac{11}{12}\right)^{19}$
$= 0.319$ (3 s.f.)

b $P(X \leqslant 4) = 0.978$ (3 s.f.)

Exam-style questions 13 page 365

1 **a**



b Using the tree diagram, there are two possibilities: blue is selected first and green second, and green is selected first and blue second.

Probability of blue and then green

$= \frac{1}{4} \times \frac{1}{3} = \frac{1}{12}$

Probability of green and then blue

$= \frac{1}{4} \times \frac{1}{3} = \frac{1}{12}$

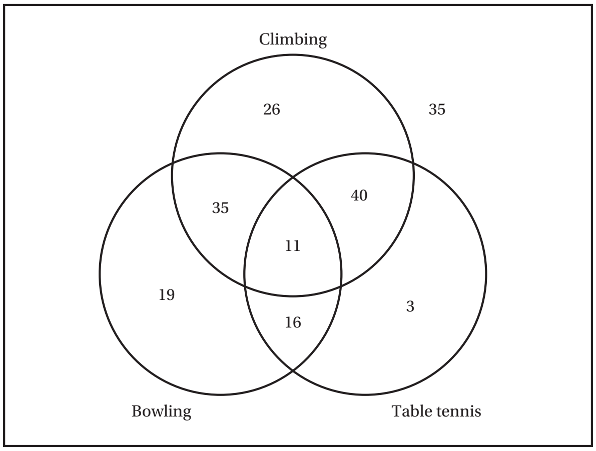
Therefore the probability of a blue kart and a green kart racing $= \frac{1}{12} + \frac{1}{12} = \frac{1}{6}$.

2 a $P(\text{male}) = \frac{140}{300} = \frac{7}{15}$

b $P(\text{male and 1 takeaway}) = \frac{40}{300} = \frac{2}{15}$

c $P(\text{female and more than 2 takeaways}) = \frac{9}{300} = \frac{3}{100}$

3 a



b 35 out of 185 do none of the activities.

Therefore the probability of a student doing none of the activities is $35 \div 185 = 0.189$ (3 s.f.)

c 19 out of 185 do bowling only.

Therefore the probability of a student doing only bowling is $19 \div 185 = 0.103$ (3 s.f.).

4 a $P(\text{All three are late}) = 0.1 \times 0.5 \times 0.25 = 0.0125$

b The probability that Jessica is not late is $1 - 0.1 = 0.9$

The probability that Demelza is not late is $1 - 0.5 = 0.5$

The probability that Kamila is not late is $1 - 0.25 = 0.75$

Therefore the probability that none of them is late is $0.9 \times 0.5 \times 0.75 = 0.3375$

$P(\text{at least one late}) = 1 - P(\text{none late})$
$= 0.6625$

c i $P(\text{both are late}) = 0.75 \times 0.5 = 0.375$

ii $P(\text{at most one}) = 1 - P(\text{both late})$
$= 1 - 0.375$
$= 0.625$

5 a $P(\text{Uche does not}) = 1 - 0.62 = 0.38$

$P(\text{Eli does not}) = 1 - 0.17 = 0.83$

$P(\text{Ellis does not}) = 1 - 0.68 = 0.32$

Therefore, $P(\text{none of them}) = 0.38 \times 0.83 \times 0.32$
$= 0.100\,928$

b $P(\text{at least one}) = 1 - P(\text{none})$
$= 1 - 0.100\,928$
$= 0.899\,072$

c There are three different possibilities:

only Uche, only Eli, and only Ellis.

$P(\text{only Uche}) = 0.62 \times 0.83 \times 0.32 = 0.164672$

$P(\text{only Eli}) = 0.38 \times 0.17 \times 0.32$
$= 0.020\,672$

$P(\text{only Ellis}) = 0.38 \times 0.83 \times 0.68$
$= 0.214\,472$

Adding the three possibilities together:
$0.164\,672 + 0.020\,672 + 0.214\,472 = 0.399\,816$

d From **part c**, $P(\text{only one}) = 0.399\,816$

There are three possibilities where only two of them hit a bull's eye: only Uche does not, only Eli does not, and only Ellis does not.

$P(\text{only Uche does not}) = 0.38 \times 0.17 \times 0.68$

$P(\text{only Eli does not}) = 0.62 \times 0.83 \times 0.68$

$P(\text{only Ellis does not}) = 0.62 \times 0.17 \times 0.32$

Therefore $P(\text{only two}) = 0.427\,584$

Therefore $P(\text{only one or two}) = 0.399\,816 + 0.427\,584$
$= 0.8274$

6 a There are 50 badges in total, therefore $n = 50$

Since the problem can be modelled as a binomial distribution, by letting X be the number of red badges, the distribution of the badges is $X \sim B(50, 0.2)$

Therefore $P(X \leqslant 15) = 0.969$ (3 s.f.)

b $P(X = 10) = {}^{50}C_{10} \times (0.2)^{10} \times (0.8)^{40} = 0.140$ (3 s.f.)

c $P(5 < X < 15) = P(5 < X \leqslant 14)$
$= P(X \leqslant 14) - P(X \leqslant 5)$
$= 0.9392\ldots - 0.04802\ldots$
$= 0.891$ (3 s.f.)

7 a $X \sim B(10, 0.21)$

Therefore $P(X = 2) = {}^{10}C_2 \times (0.21)^2 \times (0.79)^8$
$= 0.301$ (3 s.f.)

b $P(X \leqslant 3) = 0.861$ (3 s.f.)

P(at least 4) $= 1 - P(X \leqslant 3)$

$= 1 - 0.861 = 0.139$ (3 s.f.)

c $P(3 < X < 9) = P(X \leqslant 8) - P(X \leqslant 3)$

$= 0.9999\ldots - 0.86086\ldots$

$= 0.139$ (3 s.f.)

8 **a** Since each person passing her is independent of each other and there is a fixed probability of taking a leaflet with a set number of people passing, the problem can be modelled as a binomial distribution.

b In the time interval, 40 people have passed her. Therefore $n = 40$

The probability of a person taking a leaflet is 0.25. Therefore $p = 0.25$

Therefore, by letting X be the number of people taking a leaflet:

$X \sim B(40, 0.25)$

$P(X > 10) = 1 - P(X \leqslant 10)$

$= 1 - 0.583904$

$= 0.416$ (3 d.p.)

9 **a** Since each of the 15 seeds yields flowers independently of each other, with a fixed probability of yielding a red flower, the number of flowers yielded from the seeds can be modelled as a binomial distribution.

Since there are 15 seeds, $n = 15$

Since 10% of the seeds will yield red flowers, $p = 0.1$

Therefore, by letting X be the number of red flowers yielded, $X \sim B(15, 0.1)$

Therefore $P(X = 3) = {}^{15}C_3 \times (0.1)^3 \times (0.9)^{12}$

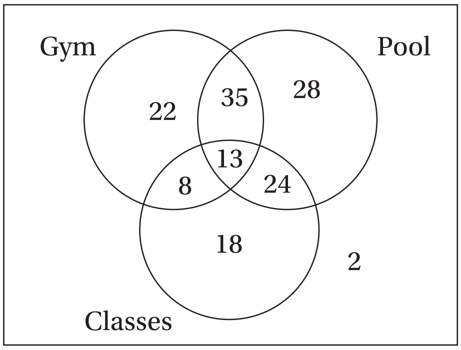
$= 0.129$ (3 s.f.)

b $P(X = 15) = {}^{15}C_{15} \times (0.1)^{15} \times (0.9)^0$

$= 0.000$

(3 s.f.) (actual answer is 1×10^{-15})

10 **a**



b P(gym only) $= \frac{22}{150}$

c P(no gym) $= 1 - \frac{78}{150} = \frac{72}{150}$

d P(gym or pool but no classes) $= 22 + 35 + 28 = \frac{85}{150}$

e P(Given uses the gym also used the pool and classes) $= \frac{56}{78}$

11

T	1	2	3	4	5
$P(T=t)$	$k(4-1) = 3k$	$k(4-2) = 2k$	$k(4-3) = k$	$k(4-4) = 0$	$k(4-5) = -1k$
	$\frac{3}{5}$	$\frac{2}{5}$	$\frac{1}{5}$	0	$-\frac{1}{5}$

$3k + 2k + k - k = 5k$

$5k = 1$

$k = \frac{1}{5}$

14 Statistical sampling and hypothesis testing

Prior knowledge page 369

1 Every member of the population has the same chance of being included in the sample.

2 Every student would be numbered, 1 to 100, and then using random number tables or random numbers on your calculator, you pick numbers until you have a sample of 30.

3 Everyone has the same chance of being selected but the sample doesn't take the proportions of individual strata into account.

Using the large data set 14.1 page 372

a
- Clean data.
- Number all the remaining data.
- Use random numbers to generate 30 for your random sample.

Random sample suggestion:

Date	Daily total rainfall (0900–0900) (mm)
10/05/87	0
11/05/87	6
20/05/87	0
28/05/87	0
01/06/87	9
06/06/87	0.2
09/06/87	1

Date	Daily total rainfall (0900–0900) (mm)
21/06/87	2.2
02/07/87	0
06/07/87	0
08/07/87	0
25/07/87	0
14/08/87	0
15/08/87	0
19/08/87	0.4
25/08/87	5.5
26/08/87	2.7
03/09/87	0.1
06/09/87	11.8
07/09/87	0.2
12/09/87	6
15/09/87	1
21/09/87	3.8
27/09/87	0
05/10/87	6.3
11/10/87	2.1
18/10/87	36.4
24/10/87	0
27/10/87	7.4
29/10/87	1

Mean = 3.4

SD = 7

Systematic sample suggestion:

$\frac{156}{30} = 5.2$, so select every 5th date from the data set,

Date	Daily total rainfall (0900–0900) (mm)
05/05/87	0
11/05/87	6
20/05/87	0
25/05/87	1.7
31/05/87	0
05/06/87	25.9
11/06/87	1.3
16/06/87	2
23/06/87	0.7
29/06/87	0.2
05/07/87	0
10/07/87	0.1
15/07/87	5.6

Date	Daily total rainfall (0900–0900) (mm)
22/07/87	0
01/08/87	1.1
06/08/87	1.8
11/08/87	4
17/08/87	0.5
23/08/87	1.1
30/08/87	0
04/09/87	13.8
11/09/87	1.9
17/09/87	0.4
22/09/87	4.2
27/09/87	0
03/10/87	14
09/10/87	11.4
14/10/87	14.9
19/10/87	12.3
24/10/87	0

Mean = 4.16

SD = 6.3

b Student's own response, comparing mean and standard deviation for samples chosen. Although the sample sizes are the same, different samples from the same population have produced different results.

c After cleaning the data, you may be left with fewer than 30 data items

Exercise 14.1A — page 373

1 **a** A sampling unit is an individual member of the population that can be sampled.

b A sampling frame is the collection of all of the sampling units.

c Sampling distribution means the type of distribution of a sample.

2 **a** So everyone in the entire target population has an equal chance of being selected and each selection is independent of each other.

b A fair proportion but not all of the cricket club supporters, for example 20% or 40% of the cricket club supporters.

c The sampling units are the cricket club supporters.

3 **a** A census is a sampling method where every member of the population is used. Consequently, it is unbiased and it gives an accurate answer.

b Since the sample is taken from a large population of 500, this sample of 5 will be adequately representative.

c By giving each individual vacuum cleaner a number, from which a random sample of 5 can be taken.

d The sampling units are the vacuum cleaners.

4 **a** The population is all teenagers, not just this sample of 242.

b The sampling distribution is the probability distribution of this sample of 242.

5 Number all of the staff, 1 to 50. The staff members with the following numbers would form the selection: 13, 15, 41, 46, 10

6 **a** The responses obtained may not be representative as these responses may come from school students of certain group(s). Bias could also derive from not obtaining many responses.

b Taking a census approach would collect the data from all the students at the school, which will accurately represent the population of these school students.

7 **a** Taking a random sample randomly selects a sample of 10 students, whereas a stratified sample approach is designed to be more representative of the whole population as the sample is aimed to represent all types of groups within the population.

b Taking a random sample randomly selects a sample of 10 students, whereas the quota sampling approach splits the population into distinct groups where a sample is taken. This quota sampling approach is generally more representative of the population than a random sample.

c Systematic sampling involves choosing subjects in a systematic (orderly or logical) way from the target population, whereas opportunity sampling is done as a matter of convenience by selecting people who are available.

Using the large data set 14.2 — page 375

a H_0: The climate has stayed the same.

H_1: The climate has changed.

2-tail as it only states 'changed'.

b Student's own selection. To be a stratified sample, it should have the following characteristics:

- Each month should be proportionally represented (May = 31/184 * 30 = 5.05 days, June = 30/184 * 30 = 4.89 days, July = 31/184 * 30 = 5.05 days, August = 31/184 * 30 = 5.05 days, September = 30/184 * 30 = 4.89 days, October = 31/184 * 30 = 5.05 days). So 5 days from each month but students should be justifying this.
- The 5 dates from each month are chosen randomly. Look for a random number selection being used.

Students need to analyse their own samples.

Look for comparisons such as:

- Max temp in 1987 was 28.2 compared to 2015 which was 23.2.
- Max rainfall in 1987 was 28.5 mm compared to 2015 which was 17.6 mm.

c There is not enough evidence to suggest climate change, only evidence to suggest that 1987 was a hotter year compared to 2015, but 2015 had less rainfall than 1987.

Exercise 14.2A — page 376

1 **a** $H_0 : p = 0.55$

b $H_1 : p < 0.55$

c 1-tail test

2 **a** $H_0 : p = 0.68$

b $H_1 : p > 0.68$

c 1-tail test

3 **a** $H_0 : p = 0.65$

b $H_1 : p > 0.65$

c 1-tail test

Exercise 14.2B — page 385

1 Critical region α lies where $\alpha < 0.025$, $\alpha > 0.975$

$P(X \leqslant 26) = 0.0194$ and $P(X \leqslant 27) = 0.0432$

Therefore one critical region lies where $X \leqslant 26$

$P(X \geqslant 37) = 0.0285$ and $P(X \geqslant 38) = 0.0079$

Therefore the other critical region lies at $X \geqslant 38$

2 Critical region α lies where $\alpha > 0.05$

$P(X \geqslant 4) = 0.121$ and $P(X \geqslant 5) = 0.0328$

Therefore the critical region lies where $X \geqslant 5$

3 Critical region α lies where $\alpha < 0.1$

$P(X \leqslant 0) = 0.0388$ and $P(X \leqslant 1) = 0.1756$

Therefore the critical region lies where $X \leqslant 0$

4 Critical region α lies where $\alpha < 0.05$

$P(X \leqslant 4) = 0.0189$ and $P(X \leqslant 5) = 0.0553$

Therefore the critical region lies where $X \leqslant 4$

5 Used $\alpha = 10\%$ at both ends, not 5%

$P(X \geqslant 9) = 0.0578 > \alpha$

$P(X \geqslant 10) = 0.0245 < \alpha$

So the critical region is $P(X \geqslant 10)$

$P(X \leqslant 2)$ is $0.1117 > \alpha$

$P(X \leqslant 1)$ is $0.0338 < \alpha$

So the critical region is $P(X \leqslant 1)$

Combined, the critical regions are $P(X \geqslant 10)$ and $P(X \leqslant 1)$

6 Less than, rather than greater than

7 Used $\alpha = 10\%$ at both ends, not 5%. Correct critical regions are $X \leqslant 1$ and $X \geqslant 9$.

8 **a** $P(X = 2) = {}^{10}C_2 \times 0.35^2 \times 0.65^8$

$= 0.176$ (3 s.f.)

b $P(X = 4) = {}^{10}C_4 \times 0.35^4 \times 0.65^6$

$= 0.238$ (3 s.f.)

c $P(X < 5) = P(X \leqslant 4)$

$= 0.752$ (3 s.f.)

d $P(X > 4) = 1 - P(X \leqslant 4)$

$= 1 - 0.7515$

$= 0.249$ (3 s.f.)

9 **a** Since there is a set number of independent lessons and a fixed probability of a student being late, the problem can be modelled as a binomial distribution.

Since there are 10 lessons, $n = 10$

Since the probability of a student being late is 0.1, $p = 0.1$

Therefore $X \sim B(10, 0.1)$

Therefore $P(X = 3) = {}^{10}C_3 \times 0.1^3 \times 0.9^7$

$= 0.0574$ (3 s.f.)

b $P(X < 3) = P(X \leqslant 2)$

$= 0.930$ (3 s.f.)

10 **a** The distribution can be defined as $X \sim B(200, 0.125)$

Therefore $P(X = 20) = 0.051$ (3 s.f.)

b $P(X > 30) = 1 - P(X \leqslant 29)$

$= 1 - 0.8327\ldots$

$= 0.1672$

c $P(20 \leqslant X \leqslant 30) = 0.878556 - 0.117297$

$= 0.761$ (3 s.f.)

11 **a** Let the digit generated be X.

Since the only possible digits greater than 5 are 6, 7, 8 and 9, $P(X > 5) = 0.4$

b Since the digits are generated independently, with a fixed probability of generating a digit greater than 5, the problem can be modelled as a binomial distribution.

Since 10 digits are generated, $n = 10$

Since there are four digits greater than 5, $p = 0.4$

Therefore, $X \sim B(10, 0.4)$

$P(X < 4) = P(X \leqslant 3)$

$= 0.382$ (3 s.f.)

12 Since there is a set number of independent recorded waiting times and a fixed probability of waiting more than 15 minutes, the problem can be modelled as a binomial distribution.

Since there are 20 recorded waiting times, $n = 20$

Since the probability of a patient having to wait more than 15 minutes is 0.3, $p = 0.3$

By letting X be the number of patients waiting more than 15 minutes, $X \sim B(20, 0.3)$

Testing the doctor's claim proposes the following hypotheses:

$H_0 : p = 0.3$

$H_1 : p < 0.3$, at 5% significance level.

Therefore the critical region α lies where $\alpha < 0.05$

For the distribution $X \sim B(20, 0.3)$, the critical region lies where $X \leqslant 2$

This is because $P(X \leqslant 2) = 0.0355$, which is less than 0.05; whereas $P(X \leqslant 3) = 0.1071$, which is greater than 0.05

Since the test value is $X = 3$, which is not in the critical region (α), there is not enough evidence to suggest that the doctor's claim is correct.

13 Since there is a set number of independent games and a fixed probability of winning, the problem can be modelled as a binomial distribution.

Since there are 8 games played, $n = 8$

Since the probability of James winning is 0.4, $p = 0.4$

By letting X be the number of games won, $X \sim B(8, 0.4)$

Testing James' claim proposes the following hypotheses:

$H_0 : p = 0.4$

$H_1 : p > 0.4$, at 5% significance level

Therefore the critical region α lies where $\alpha > 0.95$

For the distribution $X \sim B(8, 0.4)$, the critical region lies where $X \geqslant 5$

This is because $P(X \leqslant 4) = 0.8263$, which is less than 0.95, whereas $P(X \leqslant 5) = 0.9502$, which is greater than 0.95

Since the test value is $X = 6$, which does lie in the critical region, there is enough evidence to suggest that James' claim is correct – that he has improved in his game.

14 Since there is a set number of independent sweets delivered with a fixed probability of getting a red sweet, the problem can be modelled as a binomial distribution.

Since there are 20 sweets delivered, $n = 20$

Since the probability of getting a red sweet delivered is 0.7, $p = 0.7$

By letting X be the number of red sweets delivered, $X \sim B(20, 0.7)$

Testing the company's test proposes the following hypotheses:

$H_0 : p = 0.7$

$H_1 : p \neq 0.7$, at 10% significance level

Therefore the critical region α lies where

$\alpha < 0.05$ and $\alpha > 0.95$

For the distribution $X \sim B(20, 0.7)$, the critical region firstly lies where $X \leqslant 10$

This is because $P(X \leqslant 10) = 0.0480$, which is less than 0.05, whereas $P(X \leqslant 11) = 0.1133$, which is greater than 0.05

Secondly, the critical region also lies where $X > 17$

This is because $P(X \leqslant 16) = 0.8929$, which is less than 0.95, whereas $P(X \leqslant 17) = 0.9645$, which is greater than 0.95

Since the test value is $X = 10$, which does lie in the critical region, there is enough evidence to suggest that the proportion of red sweets delivered has changed.

Exam-style questions 14 — page 389

1 **a** A – quota sampling allows you to collect a sample of a suitable size without worrying about collecting a fairly representative sample.

b D – random sampling is where every member of the population would have the same chance of being selected.

c C – stratified sampling means taking a fair number from each stratum and then randomly selecting that number from within the stratum.

2 Calculating the respective proportions of students in each year:

Year 7 = $\frac{114}{600} = 0.19$

Year 8 = $\frac{132}{600} = 0.22$

Year 9 = $\frac{120}{600} = 0.2$

Year 10 = $\frac{126}{600} = 0.21$

Year 11 = $\frac{108}{600} = 0.18$

Multiplying each of these by 200 gives the number of students in each year for the sample:

Year 7 = 38 students

Year 8 = 44 students

Year 9 = 40 students

Year 10 = 42 students

Year 11 = 36 students

3 Stan's proposition can be modelled as a binomial distribution.

By letting X be the number of correctly predicted passes, $X \sim B(50, 0.54)$

Stan's claim can be tested with the following hypotheses:

$H_0 : p = 0.54$ (Stan's claim is correct)

$H_1 : p > 0.54$ (Stan's claim is incorrect)

The critical region α lies where $\alpha \geqslant 0.95$

The critical region lies where $X \geqslant 34$, as $P(X \leqslant 31) = 0.8998$ and $P(X \leqslant 32) = 0.9418$ and $P(X \leqslant 32) = 0.9687$.

Since the test value is $X = 31$, which is not in the critical region, there is not enough evidence to suggest that Stan's claim is incorrect (accept H_0).

4 **a** The problem can be modelled as a binomial distribution such that $X \sim B(30, 0.40)$, where X is the number of clients who bought a protein shake.

The test can be tested under the following hypotheses:

$H_0 : p = 0.40$

$H_1 : p \neq 0.40$

The critical region α lies where $\alpha < 0.025$

Since $P(X \leqslant 11) = 0.4311$

Since 11 clients bought a protein shake, the test value is $X = 11$

Since $X = 11$ does not lie in the critical region, there is enough evidence to suggest that there has been no change in the proportion of customers buying a protein shake.

b The problem can be modelled as a binomial distribution such that $X \sim B(100, 0.35)$, where X is the number of customers who bought fruit.

The test can be tested under the following hypotheses:

$H_0 : p = 0.35$

$H_1 : p > 0.35$

$X = 18$

$P(X \geqslant 18) = 0.99995$ which does not lie in the critical region

so there is not enough evidence to suggest that the manager's claim is correct

5 **a** The critical region is the range of values which would lead you to reject the null hypothesis.

b The significance level is the probability of rejecting the null hypothesis when it is in fact true.

6 Since there is a set number of independent questions and a fixed probability of getting a question correct, the problem can be modelled as a binomial distribution.

By letting X be the number of questions answered correctly, $X \sim B(20\ 0.2)$

The hypotheses are as follows:

$H_0 : p = 0.2$

$H_1 : p \neq 0.2$

The critical region α lies between $\alpha > 0.025$ and $\alpha > 0.975$

$P(X \leqslant 0) = 0.0115$ and $P(X \leqslant 1) = 0.0692$, so the critical region lies where $X \leqslant 0$

$P(X \leqslant 7) = 0.9679$ and $P(X \leqslant 8) = 0.9900$, so the critical region lies where $X > 8$ or $X \geqslant 9$.

Since the learner driver got six questions correct ($X = 6$), which does not lie in the critical region, there is *not* enough evidence to suggest that Stan's claim was correct.

7 **a** The population would be all the foods produced of this particular variety.

b A possible sampling frame would be a list or barcodes of all the meals produced by the machine in a particular time frame.

c Using a random sample – students own description of this process using random number tables or another reliable process.

8 Let p be the probability that a dice lands on a six.

$H_0: p = \frac{1}{6}$

$H_1: p > \frac{1}{6}$

Significance level = 10%

Let X be the number of times a six was recorded out of the 20 rolls.

$X \sim B(20, \frac{1}{6})$

$P(X \geqslant 6) = 1 - P(X \leqslant 5)$

$1 - 0.8982 = 0.1018$

$0.1018 > 10\%$ so accept null, hypothesis not enough evidence to suggest that the dice is biased towards 6.

9 **a** $P(X \geqslant 24) = 1 - P(X \leqslant 23)$

$1 - 0.729 = 0.271$

b $P(X \geqslant 24) = 1 - P(X \leqslant 23)$

$1 - 0.993 = 0.007$

c Let p be the probability that a child improves it spellings.

H_0: $p = 90\%$ company states 90% improvement

H_1: $p < 90\%$ school claims the improvement rate is not that high

Significance level = 5%

d Let X be the number of children in the class.

$X \sim B(25, 0.90)$

$P(X \leqslant 19) = 0.0334$

$P(X \leqslant 20) = 0.09799$

$P(X \leqslant 21) = 0.2364$

$0.2364 > 0.05$

so, if the number of children is 20 or more, the null hypothesis will be rejected and evidence will suggest the alternate hypothesis is true.

15 Kinematics

Prior knowledge page 392

1 **a** $v^2 = u^2 + 2as$

$v^2 = 40^2 + 2 \times 18 \times 49$

$v^2 = 3364$

$v = 58$

b $s = \frac{v^2 - v^2}{2a}$

$s = \frac{148^2 - 48^2}{2 \times 9.8}$

$s = 1000$

2 $\frac{1}{2} \times (19 + 34) \times 22 = 583\ \text{cm}^2$

3 **a** $4(T + 2) = 3T + 14$

$4T + 8 = 3T + 14$

$T = 6$

b $2T^2 - 5T - 12 = 0$

$(2T + 3)(T - 4) = 0$

$T = 4$ or -1.5

4 **a** $f'(x) = 12x - 7$

b $\int f(x)\ dx = 2x^3 - \frac{7}{2}x^2 + 8x + c$

Exercise 15.1A page 394

1 Overall distance travelled = 30 + 20 = 50 m

Displacement from original position
= –30 + 20 = –10 m

2 Overall distance travelled by the ball
= (40 – 15) + 40 = 25 + 40 = 65 m

Displacement of the ball from original position
= –15 m

3 **a** 32 + 24 = 56 m

b By walking west then south, the boy is tracing out the two perpendicular sides of a right-angled triangle.

Hypotenuse = $\sqrt{32^2 + 24^2}$ = 40 m

c Speed = distance ÷ time = 56 ÷ 28 = 2 m s^{-1}

4 **a** 2(7 + 4) = 22 m

b Since the snail returns to its initial position, the overall displacement = 0 m.

Exercise 15.2A page 399

1 **a** $s = ut + \frac{1}{2}at^2$

$s = 70 \times 10 + \frac{1}{2} \times -3 \times 10^2$

$s = 550\ \text{m}$

b $s = \left(\frac{u + v}{2}\right)t$

$s = \left(\frac{15+29}{2}\right) \times 9$

$s = 198\ \text{m}$

c $v = u + at$

$a = \frac{v-u}{t}$

$a = \frac{38-3}{7}$

$a = 5\ \text{m s}^{-2}$

d $v^2 = u^2 + 2as$

$s = \frac{v^2-u^2}{2a}$

$s = \frac{28^2-22^2}{2\times6}$

$s = 25\ \text{m}$

e $s = vt - \frac{1}{2}at^2$

$0 = -28t - \frac{1}{2} \times -7t^2$

$\frac{7}{2}t^2 = 28t$

$t^2 = 8t$

$t = 0$ or $8\ \text{s}$

2 **a** $u = 24\ \text{m s}^{-1}$, $a = 5\ \text{m s}^{-2}$, $s = 10\ \text{m}$

$v^2 = u^2 + 2as$

$v^2 = 24^2 + 2 \times 5 \times 10 = 676$

$v = 26\ \text{m s}^{-1}$

b $v = u + at$

$t = \frac{v-u}{a}$

$t = \frac{26-24}{5}$

$t = 0.4\ \text{s}$

3 **a** $u = 60\ \text{m s}^{-1}$, $a = -8\ \text{m s}^{-2}$, $t = 10\ \text{s}$

$v = u + at$

$v = 60 - 8 \times 10$

$v = -20\ \text{m s}^{-1}$

b $v = 0\ \text{m s}^{-1}$

$v = u + at$

$t = \frac{v-u}{a}$

$t = \frac{0-60}{-8}$

$t = 7.5\ \text{s}$

4 **a** $u = 0\ \text{m s}^{-1}$, $t = 40\ \text{s}$, $a = 0.5\ \text{m s}^{-2}$

$v = u + at$

$v = 0 + 0.5 \times 40$

$v = 20\ \text{m s}^{-1}$

b $s = ut + \frac{1}{2}at^2$

$s = 0 \times 40 + \frac{1}{2} \times 0.5 \times 40^2$

$s = 400\ \text{m}$

5 **a** $u = 50\,\text{m s}^{-1}$, $s = 1500\,\text{m}$, $v = 0\,\text{m s}^{-1}$

$v^2 = u^2 + 2as$

$a = \frac{v^2 - u^2}{2s}$

$a = \frac{0^2 - 50^2}{2 \times 1500}$

$a = -\frac{5}{6}\,\text{m s}^{-2}$

$v = u + at$

$t = \frac{v - u}{a}$

$t = \frac{0 - 50}{-\frac{5}{6}}$

$t = 60\,\text{s}$

b Deceleration is constant, etc.

6 **a** $u = 0\,\text{m s}^{-1}$, $a = 4\,\text{m s}^{-2}$, $s = 128\,\text{m}$

$s = ut + \frac{1}{2}at^2$

$128 = 0 \times t + \frac{1}{2} \times 4 \times t^2$

$128 = 2t^2$

$t^2 = 64$

$t = 8\,\text{s}$

b $u = 180\,\text{m s}^{-1}$, $a = -3\,\text{m s}^{-2}$, $s = 0\,\text{m}$

$s = ut + \frac{1}{2}at^2$

$0 = 180 \times t + \frac{1}{2} \times -3 \times t^2$

$\frac{3}{2}t^2 - 180t = 0$

$t^2 - 120t = 0$

$t(t - 120) = 0$

$t = 0$ or $120\,\text{s}$

It takes 2 minutes.

c $u = 5\,\text{m s}^{-1}$, $a = 3\,\text{m s}^{-2}$, $s = 84\,\text{m}$

$84 = 5T + \frac{1}{2} \times 3T^2$

$\frac{3}{2}T^2 + 5T - 84 = 0$

$3T^2 + 10T - 168 = 0$

$(3T + 28)(T - 6) = 0$

$T = 6\,\text{s}$

7 $a = 0.25\,\text{m s}^{-2}$, $s = 930\,\text{m}$, $t = 60\,\text{s}$

$930 = u \times 60 + \frac{1}{2} \times 0.25 \times 60^2$

$930 = 60u + 450$

$60u = 480$

$u = 8\,\text{m s}^{-1}$

$v = u + at$

$v = 8 + 0.25 \times 60$

$v = 23\,\text{m s}^{-1}$

8 $60\,\text{km h}^{-1} = 60\,000$ metres per hour $= \frac{50}{3}\,\text{m s}^{-1}$

$u = 0\,\text{m s}^{-1}$, $v = \frac{50}{3}\,\text{m s}^{-1}$, $t = 8\,\text{s}$

$s = \left(\frac{u + v}{2}\right)t$

$s = \left(\frac{0 + \frac{50}{3}}{2}\right) \times 8$

$s = \frac{200}{3}\,\text{m}$

a $200 - \frac{200}{3} = \frac{400}{3}\,\text{m}$

$s = ut$

$\frac{400}{3} = \frac{50}{3}t$

$t = 8$

Total time $= 8 + 8 = 16\,\text{s}$

b $1000 - \frac{200}{3} = \frac{2800}{3}\,\text{m}$

$s = ut$

$\frac{2800}{3} = \frac{50}{3}t$

$t = 56$

Total time $= 8 + 56 = 64\,\text{s}$

Exercise 15.2B page 401

1 **a** PQ: $t = 5\,\text{s}$, $s = 100\,\text{m}$

$s = ut + \frac{1}{2}at^2$

$100 = 5u + \frac{25}{2}a$

$40 = 2u + 5a$ ①

PR: $t = 20\,\text{s}$, $s = 820\,\text{m}$

$s = ut + \frac{1}{2}at^2$

$820 = 20u + 200a$

$82 = 2u + 20a$ ②

Subtract ① from ②.

$42 = 15a$

$a = 2.8\,\text{m s}^{-2}$

b PQ: $t = 5\,\text{s}$, $s = 100\,\text{m}$, $a = 2.8\,\text{m s}^{-2}$

Could find v from t, s and a, but simpler to find and use u.

Substitute in ①.

$40 = 2u + 5 \times 2.8$

$40 = 2u + 14$

$26 = 2u$

$u = 13\,\text{m s}^{-1}$

$v = u + at$

$v = 13 + 2.8 \times 5$

$v = 27\,\text{m s}^{-1}$

c PS: $u = 13\,\text{m s}^{-1}$, $a = 2.8\,\text{m s}^{-2}$,

$s = 830 + 820 = 1650\,\text{m}$

$v^2 = u^2 + 2as$

$v^2 = 13^2 + 2 \times 2.8 \times 1650$

$v^2 = 9409$

$v = 97\text{ m s}^{-1}$

$v = u + at$

$t = \frac{v-u}{a}$

$t = \frac{97-13}{2.8}$

$t = 30\text{ s}$

Time from Q to S = 30 – 20 = 10 s

2 **a** $u = 20\text{ m s}^{-1}$, $a = -4\text{ m s}^{-2}$, $v = 0\text{ m s}^{-1}$

$v^2 = u^2 + 2as$

$s = \frac{v^2-u^2}{2a}$

$s = \frac{0^2-20^2}{2\times-4}$

$s = 50\text{ m}$

Since the debris is 80 m away, he will stop 30 m from the debris.

b $u = 20\text{ m s}^{-1}$, $a = 2\text{ m s}^{-2}$, $t = 2\text{ s}$

$v = u + at$

$v = 20 + 2 \times 2$

$v = 24\text{ m s}^{-1}$

$s = ut + \frac{1}{2}at^2$

$s = 20 \times 2 + \frac{1}{2} \times 2 \times 2^2$

$s = 44\text{ m}$

Distance from debris = 80 – 44 = 36 m

$u = 24\text{ m s}^{-1}$, $a = -4\text{ m s}^{-2}$, $s = 36\text{ m}$

$v^2 = u^2 + 2as$

$v^2 = 24^2 + 2 \times -4 \times 36$

$v^2 = 288$

$v = \sqrt{288} = 17.0\text{ m s}^{-1}$

3 Let the displacement of A from its starting point be s_A m when $t = T$ s.

Let the displacement of B from its starting point be s_B m when $t = T$ s.

For A: $s = s_A$ m, $t = T$ s, $u = 0\text{ m s}^{-1}$, $a = 6\text{ m s}^{-2}$

$s = ut + \frac{1}{2}at^2$

$s_A = 0 \times T + \frac{1}{2} \times 6 \times T^2$

$s_A = 3T^2$

For B: $s = s_B$ m, $t = T$ s, $u = 0\text{ m s}^{-1}$, $a = -2\text{ m s}^{-2}$

$s = ut + \frac{1}{2}at^2$

$s_B = 0 \times T + \frac{1}{2} \times -2 \times T^2$

$s_B = -T^2$

Since B started 100 m from A:

$s_A - s_B = 100$

$3T^2 - (-T^2) = 100$

$4T^2 = 100$

$T^2 = 25$

$T = 5$

a When $T = 5$, $s_A = 3T^2 = 3 \times 5^2 = 75\text{ m}$

b $T = 5s$

c For A: $t = 5\text{ s}$, $u = 0\text{ m s}^{-1}$, $a = 6\text{ m s}^{-2}$

$v = u + at$

$v = 0 + 6 \times 5$

$v = 30\text{ m s}^{-1}$

For B: $t = 5\text{ s}$, $u = 0\text{ m s}^{-1}$, $a = -2\text{ m s}^{-2}$

$v = u + at$

$v = 0 - 2 \times 5$

$v = -10\text{ m s}^{-1}$

4 $u = 13\text{ m s}^{-1}$, $a = -2\text{ m s}^{-2}$, $s = \pm 30\text{ m}$

$s = ut + \frac{1}{2}at^2$

When $s = +30$ m:

$30 = 13 \times t + \frac{1}{2} \times -2 \times t^2$

$t^2 - 13t + 30 = 0$

$(t - 3)(t - 10) = 0$

$t = 3\text{ s}$ or $t = 10\text{ s}$

When $s = -30$ m:

$-30 = 13 \times t + \frac{1}{2} \times -2 \times t^2$

$t^2 - 13t - 30 = 0$

$(t - 15)(t + 2) = 0$

$t = 15\text{ s}$ or $t = -2\text{ s}$

Times when ball is 30 m from its starting point are 3 s, 10 s and 15 s.

5 For J: $u = 2\text{ m s}^{-1}$, $a = 1\text{ m s}^{-2}$, $s = 30\text{ m}$

$s = ut + \frac{1}{2}at^2$

$30 = 2 \times t + \frac{1}{2} \times 1 \times t^2$

$t^2 + 4t - 60 = 0$

$(t + 10)(t - 6) = 0$

$t = 6\text{ s}$

For K: $u = 0\text{ m s}^{-1}$, $a = 0.8\text{ m s}^{-2}$, $v = 4\text{ m s}^{-1}$

$v = u + at$

$4 = 0 + 0.8t$

$t = 5\text{ s}$

For L: $u = 3\text{ m s}^{-1}$, $a = 0.5\text{ m s}^{-2}$, $v = 7\text{ m s}^{-1}$

$v = u + at$

$7 = 3 + 0.5t$

$t = 8\text{ s}$

When L has a velocity of 7 m s^{-1}, J has been moving for 19 s (6 + 5 + 8) and K for 13 s (5 + 8).

For J: $u = 2\text{ m s}^{-1}$, $a = 1\text{ m s}^{-2}$, $t = 19\text{ s}$

$s = ut + \frac{1}{2}at^2$

$s = 2 \times 19 + \frac{1}{2} \times 1 \times 19^2$

$s = 218.5\text{ m}$

For K: $u = 0\,\text{m s}^{-1}$, $a = 0.8\,\text{m s}^{-2}$, $t = 13\,\text{s}$

$s = ut + \frac{1}{2}at^2$

$s = 0 \times 13 + \frac{1}{2} \times 0.8 \times 13^2$

$s = 67.6\,\text{m}$

Distance between J and K $= 218.5 - 67.6$
$= 150.9\,\text{m}$

6 **a** Let $U\,\text{m s}^{-1}$ be the velocity of the object at E and $V\,\text{m s}^{-1}$ be the velocity of the object at F.

FG: $u = V\,\text{m s}^{-1}$, $t = 3\,\text{s}$, $s = 66\,\text{m}$

$s = ut + \frac{1}{2}at^2$

$66 = 3V + \frac{1}{2} \times a \times 3^2$

$44 = 2V + 3a$ ①

EF: $v = V\,\text{m s}^{-1}$, $t = 7\,\text{s}$, $u = U\,\text{m s}^{-1}$

$v = u + at$

$V = U + 7a$ ②

Substitute V from ② into ①.

$44 = 2(U + 7a) + 3a$

$44 = 2U + 17a$ ③

EH: $t = 14\,\text{s}$, $v = 0\,\text{m s}^{-1}$, $u = U\,\text{m s}^{-1}$

$v = u + at$

$0 = U + 14a$ ④

Multiply ④ by 2.

$0 = 2U + 28a$ ⑤

Subtract ③ from ⑤.

$-44 = 11a$

$a = -4\,\text{m s}^{-2}$

b Substitute $a = -4$ into ④.

$0 = U + 14 \times -4$

$U = 56\,\text{m s}^{-1}$

7 **a** Let T s be the time taken between the garage (G) and the restaurant (R).

Hence the time taken between the restaurant and the library (L) is $2T$ s.

Let $V\,\text{m s}^{-1}$ be the velocity at the restaurant.

RL: $v = 9\,\text{m s}^{-1}$, $a = 0.15\,\text{m s}^{-2}$, $s = 240\,\text{m}$, $t = 2T\,\text{s}$

$s = vt - \frac{1}{2}at^2$

$240 = 9(2T) - \frac{1}{2} \times 0.15 \times (2T)^2$

$240 = 18T - 0.3T^2$

$0.3T^2 - 18T + 240 = 0$

$T^2 - 60T + 800 = 0$

$(T - 20)(T - 40) = 0$

$T = 20 \text{ or } 40$

RL: $v = 9\,\text{m s}^{-1}$, $a = 0.15\,\text{m s}^{-2}$, $t = 2T\,\text{s}$, $u = V\,\text{m s}^{-1}$

$v = u + at$

If $T = 20$: $9 = V + 0.15 \times 40$, so $V = 3\,\text{m s}^{-1}$

If $T = 40$: $9 = u + 0.15 \times 80$, so $V = -3\,\text{m s}^{-1}$

Since the cyclist is not cycling backwards, $T = 20\,\text{s}$ and $V = 3\,\text{m s}^{-1}$

GL: $v = 9\,\text{m s}^{-1}$, $a = 0.15\,\text{m s}^{-2}$, $t = 60\,\text{s}$

$s = vt - \frac{1}{2}at^2$

$s = 9 \times 60 - \frac{1}{2} \times 0.15 \times 60^2$

$s = 270\,\text{m}$

Alternatively:

GR: $v = 3\,\text{m s}^{-1}$, $t = 20\,\text{s}$, $a = 0.15\,\text{m s}^{-2}$

$s = vt - \frac{1}{2}at^2$

$s = 3 \times 20 - \frac{1}{2} \times 0.15 \times 20^2$

$s = 30\,\text{m}$

Total distance $= 30 + 240 = 270\,\text{m}$

Exercise 15.3A **page 405**

1 $u = 0\,\text{m s}^{-1}$, $a = 9.8\,\text{m s}^{-2}$, $t = 1.8\,\text{s}$

$s = ut + \frac{1}{2}at^2$

$s = 0 \times t + \frac{1}{2} \times 9.8 \times 1.8^2$

$s = 15.9\,\text{m}$

2 $u = 0\,\text{m s}^{-1}$, $a = 9.8\,\text{m s}^{-2}$, $s = 1.2\,\text{m}$

$s = ut + \frac{1}{2}at^2$

$1.2 = 0 \times t + \frac{1}{2} \times 9.8 \times t^2$

$4.9t^2 = 1.2$

$t^2 = \frac{1.2}{4.9}$

$t = \sqrt{\frac{1.2}{4.9}} = 0.495\,\text{s}$

3 $u = 7\,\text{m s}^{-1}$, $a = 9.8\,\text{m s}^{-2}$, $v = 15\,\text{m s}^{-1}$

$v^2 = u^2 + 2as$

$s = \frac{v^2 - u^2}{2a}$

$s = \frac{15^2 - 7^2}{2 \times 9.8}$

$s = 8.98\,\text{m}$

4 **a** $u = 0\,\text{m s}^{-1}$, $a = 9.8\,\text{m s}^{-2}$, $s = 48\,\text{m}$

$v^2 = u^2 + 2as$

$v^2 = 0^2 + 2 \times 9.8 \times 48$

$v^2 = 940.8$

$v = \sqrt{940.8} = 30.7\,\text{m s}^{-1}$

b $v = u + at$

$t = \frac{v - u}{a}$

$t = \frac{30.7 - 0}{9.8}$

$t = 3.13\,\text{s}$

5 $u = 39.2\,\text{m s}^{-1}$, $a = -9.8\,\text{m s}^{-2}$

a $t = 10\,\text{s}$

$s = ut + \frac{1}{2}at^2$

$s = 39.2 \times 10 + \frac{1}{2} \times -9.8 \times 10^2$

$s = -98\text{ m}$

The cliff is 98 m high.

b $v = 0\text{ m s}^{-1}$

$v^2 = u^2 + 2as$

$s = \frac{v^2 - u^2}{2a}$

$s = \frac{0^2 - 39.2^2}{2 \times -9.8}$

$s = 78.4\text{ m}$

Maximum height = 98 + 78.4 = 176.4 m

c $v = 0\text{ m s}^{-1}$

$v = u + at$

$t = \frac{v - u}{a}$

$t = \frac{0 - 39.2}{-9.8}$

$t = 4\text{ s}$

d $t = 10\text{ s}$

$v = u + at$

$v = 39.2 - 9.8 \times 10$

$v = -58.8$

Speed $= 58.8\text{ m s}^{-1}$

e $s = 0\text{ m}$

$s = ut + \frac{1}{2}at^2$

$0 = 39.2 \times t + \frac{1}{2} \times -9.8 \times t^2$

$0 = 39.2t - 4.9t^2$

$0 = 4.9t(8 - t)$

$t = 0\text{ s}$ or 8 s

The rocket takes 8 s to return to same height.

6 $u = 25\text{ m s}^{-1}$, $a = -9.8\text{ m s}^{-2}$

a $s = 0\text{ m}$

$s = ut + \frac{1}{2}at^2$

$0 = 25 \times t + \frac{1}{2} \times -9.8 \times t^2$

$0 = 25t - 4.9t^2$

$0 = t(25 - 4.9t)$

$t = 0\text{ s}$ or $\frac{25}{4.9} = 5.10\text{ s}$

b $v = 0$

$v^2 = u^2 + 2as$

$s = \frac{v^2 - u^2}{2a}$

$s = \frac{0^2 - 25^2}{2 \times -9.8}$

$s = 31.9\text{ m}$

c $s = \frac{3}{4} \times 31.9 = 23.92\text{ m}$

$v^2 = u^2 + 2as$

$v^2 = 25^2 + 2 \times -9.8 \times 23.92$

$v^2 = 156.25$

$v = \sqrt{156.25} = 12.5\text{ m s}^{-1}$

$v = u + at$

$12.5 = 25 - 9.8t$

$9.8t = 12.5$

$t = \frac{12.5}{9.8} = 1.28\text{ s}$

d Maximum height is an overestimate because of the effect of air resistance.

e No difference: according to equations, mass has no effect.

7 $s = -11.4\text{ m}$, $a = -9.8\text{ m s}^{-2}$, $t = 4.3\text{ s}$

$s = ut + \frac{1}{2}at^2$

$-11.4 = u \times 4.3 + \frac{1}{2} \times -9.8 \times 4.3^2$

$-11.4 = 4.3u - 90.601$

$4.3u = 79.201$

$u = 18.4\text{ m s}^{-1}$

$v = 0\text{ m s}^{-1}$, $a = -9.8\text{ m s}^{-2}$, $u = 18.4\text{ m s}^{-1}$, $t = T\text{ s}$

$v = u + at$

$0 = 18.4 - 9.8T$

$T = \frac{18.4}{9.8} = 1.88\text{ s}$

8 $u = 42\text{ m s}^{-1}$, $a = -9.8\text{ m s}^{-2}$, $s = 87.5\text{ m}$

$s = ut + \frac{1}{2}at^2$

$87.5 = 42 \times t + \frac{1}{2} \times -9.8 \times t^2$

$4.9t^2 - 42t + 87.5 = 0$

$t = \frac{42 \pm \sqrt{(-42)^2 - 4 \times 4.9 \times 87.5}}{2 \times 4.9}$

$t = \frac{25}{7}$ or 5

The ball is more than 87.5 m above the ground for $5 - \frac{25}{7} = \frac{10}{7}$ s.

Exercise 15.3B page 407

1 a $u = 0\text{ m s}^{-1}$, $a = g\text{ m s}^{-2}$, $s = h\text{ m}$, $t = t\text{ s}$

$s = ut + \frac{1}{2}at^2$

$h = 0 \times t + \frac{1}{2}gt^2$

$h = \frac{1}{2}gt^2$

$2h = gt^2$

$t^2 = \frac{2h}{g}$

$t = \sqrt{\frac{2h}{g}}$

b $u = 0\text{ m s}^{-1}$, $a = 3g\text{ m s}^{-2}$, $s = 12h\text{ m}$, $t = t\text{ s}$

$s = ut + \frac{1}{2}at^2$

$12h = 0 \times t + \frac{1}{2} \times 3gt^2$

$12h = \frac{3}{2}gt^2$

$8h = gt^2$

$t^2 = \frac{8h}{g}$

$t = 2\sqrt{\frac{2h}{g}}$

It would take 2 times as long.

2 Let the height of the castle be H m and the time taken by the dropped stone be T s.

Dropped stone:
$u = 0\,\text{m s}^{-1}$, $a = 9.8\,\text{m s}^{-2}$, $s = H\,\text{m}$, $t = T\,\text{s}$

$s = ut + \frac{1}{2}at^2$

$H = 0 \times T + \frac{1}{2} \times 9.8T^2$

$H = 4.9T^2$ ①

Thrown stone:
$u = 14\,\text{m s}^{-1}$, $a = 9.8\,\text{m s}^{-2}$, $s = H\,\text{m}$, $t = (T - 1)\,\text{s}$

$s = ut + \frac{1}{2}at^2$

$H = 14 \times (T - 1) + \frac{1}{2} \times 9.8(T - 1)^2$

$H = 14(T - 1) + 4.9(T^2 - 2T + 1)$

$H = 14T - 14 + 4.9T^2 - 9.8T + 4.9$ ②

Put ① = ②.

$4.9T^2 = 14T - 14 + 4.9T^2 - 9.8T + 4.9$

$0 = 14T - 14 - 9.8T + 4.9$

$4.2T = 9.1$

$T = \frac{9.1}{4.2} = 2.17\,\text{s}$

$H = 4.9T^2$

$H = 4.9 \times 2.17^2$

$H = 23.0\,\text{m}$

3 The ball is travelling in the opposite direction to the direction in which it was launched when it hits the ground, so its velocity is negative. Sawda's answer is correct.

Katie's solution is more efficient but she needs to ensure she chooses the correct sign for v.

4 a $u = 70\,\text{m s}^{-1}$, $a = -10\,\text{m s}^{-2}$, $t = T\,\text{s}$, $s > 165\,\text{m}$

$70T - 5T^2 > 165$

$0 > 5T^2 - 70T + 165$

$5T^2 - 70T + 165 < 0$

$T^2 - 14T + 33 < 0$

b $(T - 3)(T - 11) < 0$

$3 < T < 11$

The object is more than 165 m above the ground for 8 seconds.

5 Pulled by rope: $u = 0\,\text{m s}^{-1}$, $a = 0.8\,\text{m s}^{-2}$, $t = 4\,\text{s}$

$v = u + at$

$v = 0 + 0.8 \times 4$

$v = 3.2\,\text{m s}^{-1}$

$s = ut + \frac{1}{2}at^2$

$s = 0 \times 4 + \frac{1}{2} \times 0.8 \times 4^2$

$s = 6.4\,\text{m}$

After rope snaps:
$u = 3.2\,\text{m s}^{-1}$, $a = -9.8\,\text{m s}^{-2}$, $s = -6.4\,\text{m}$

$v^2 = u^2 + 2as$

$v^2 = 3.2^2 + 2 \times -9.8 \times -6.4$

$v^2 = 135.68$

$v = \sqrt{135.68} = -11.6\,\text{m s}^{-1}$

$v = -11.6\,\text{m s}^{-1}$, $u = 3.2\,\text{m s}^{-1}$, $a = -9.8\,\text{m s}^{-2}$

$v = u + at$

$-11.6 = 3.2 - 9.8t$

$9.8t = 14.8$

$t = \frac{14.8}{9.8} = 1.52\,\text{s}$

6 Halfway down the building is 18.5 m (37 ÷ 2).

Time for dropped stone to hit the ground:
$u = 0\,\text{m s}^{-1}$, $a = 9.8\,\text{m s}^{-2}$, $s = 37\,\text{m}$

$s = ut + \frac{1}{2}at^2$

$37 = 0 \times t + \frac{1}{2} \times 9.8t^2$

$t^2 = \frac{37}{4.9} = 7.55$

$t = \sqrt{7.55} = 2.75\,\text{s}$

Time for dropped stone to get halfway:
$u = 0\,\text{m s}^{-1}$, $a = 9.8\,\text{m s}^{-2}$, $s = 18.5\,\text{m}$

$s = ut + \frac{1}{2}at^2$

$18.5 = 0 \times t + \frac{1}{2} \times 9.8t^2$

$t^2 = \frac{18.5}{4.9} = 3.78$

$t = \sqrt{3.78} = 1.94\,\text{s}$

The thrown stone takes 0.805 s to get to the ground (2.75 – 1.94).

Thrown stone: $a = 9.8\,\text{m s}^{-2}$, $s = 37\,\text{m}$, $t = 0.805\,\text{s}$

$s = ut + \frac{1}{2}at^2$

$37 = u \times 0.805 + \frac{1}{2} \times 9.8 \times 0.805^2$

$37 = 0.805u + 3.174$

$33.83 = 0.805u$

$u = 42.0\,\text{m s}^{-1}$

7 a Minimum value of U:

$v = 0\,\text{m s}^{-1}$, $s = 40\,\text{m}$, $a = -9.8\,\text{m s}^{-2}$

$v^2 = u^2 + 2as$

$0 = U^2 + 2 \times -9.8 \times 40$

$U^2 = 784$

$U = \sqrt{784} = 28\,\text{m s}^{-1}$

Maximum value of U:

$v = 5\,\text{m s}^{-1}$, $s = 40\,\text{m}$, $a = -9.8\,\text{m s}^{-2}$

$5^2 = u^2 + 2as$

$25 = U^2 + 2 \times -9.8 \times 40$

$U^2 = 809$

$U = \sqrt{809} = 28.4\,\text{m s}^{-1}$

b $u = 5\text{ m s3}^{-1}$, $v = -5\text{ m s}^{-1}$, $a = -9.8\text{ m s}^{-2}$

$v = u + at$

$-5 = 5 - 9.8t$

$9.8t = 10$

$t = \frac{10}{9.8} = 1.02\text{ s}$

8 Student's own proof

Exercise 15.4A page 410

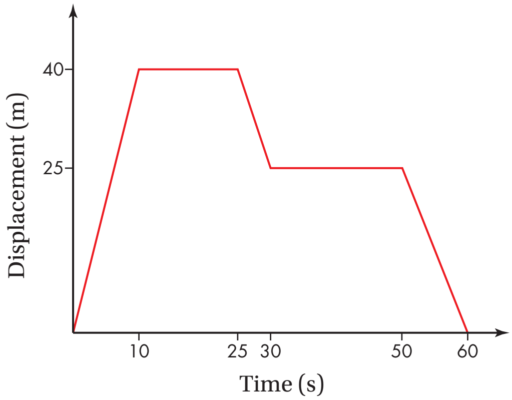
1 Stage 1 (0 s–10 s): velocity = 4 m s^{-1}, time = 10 s, displacement = $4 \times 10 = 40$ m

Stage 2 (10 s–25 s): stationary for 15 s

Stage 3 (25 s – 30 s): velocity = -3 m s^{-1}, displacement = -15 m, time = $-15 \div -3 = 5$ s

Stage 4 (30 s–50 s): stationary for 20 s

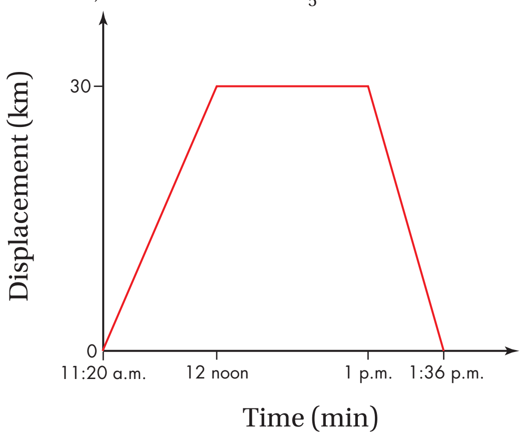
Stage 5 (50 s–60 s): velocity = -2.5 m s^{-1}, displacement = -25 m, time = $-25 \div -2.5 = 10$ s



2 **a** Stage 1: velocity = 45 km h^{-1}, time = 40 min; since 40 min = $\frac{2}{3}$ h, displacement = 30 km

Stage 2: stationary for 60 min

Stage 3: velocity = -50 km h^{-1}, displacement = -30 km; time = $-30 \div -50 = \frac{3}{5}$ h = 36 min



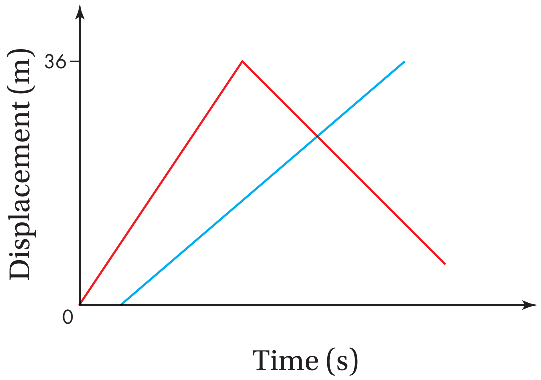
b 136 minutes after 11:20 a.m. is 1:36 p.m.

3 **a** First particle:

Stage 1 (0 s–12 s): velocity = 3 m s^{-1}, time = 12 s, displacement = $3 \times 12 = 36$ m

Stage 2 (12 s–27 s): velocity = -2 m s^{-1}, time = 15 s, displacement = $-2 \times 15 = -30$ m

b Second particle: velocity = 1.6 m s^{-1}, starting three seconds later



c The graphs cross whilst the first particle is returning.

Equation of line for first particle:

$d - 36 = -2(t - 12)$

$2t + d = 60$ ①

Equation of line for second particle:

$d - 0 = 1.6(t - 3)$

$d = 1.6t - 4.8$ ②

Substitute for d in ①.

$2t + 1.6t - 4.8 = 60$

$3.6t = 64.8$

$t = 18\text{ s}$

$d = 1.6 \times 18 - 4.8 = 24\text{ m}$

4 **a** Car A:

Stage 1: velocity = 75 km h^{-1}, time = 20 min; since 20 min = $\frac{1}{3}$ h, displacement = 25 km

Stage 2: stationary for 10 min

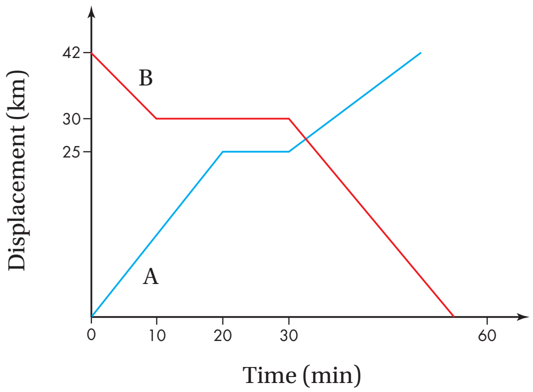
Stage 3: velocity = 51 km h^{-1}, displacement = 17 km; time = $17 \div 51 = \frac{1}{3}$ h = 20 min

Car B:

Stage 1: velocity = -72 km h^{-1}, time = 10 min; since 10 min = $\frac{1}{6}$ h, displacement = -12 km

Stage 2: stationary for 20 min

Stage 3: velocity = -72 km h^{-1}, displacement = -30 km; time = $-30 \div -72 = \frac{5}{12}$ h = 25 min



b The graphs cross whilst both cars are on their final stages.

Gradients are given as kilometres per minute.

Equation of line for first car:

$d - 25 = 0.85(t - 30)$

$d = 0.85t - 0.5$ ①

Equation of line for second car:

$d - 30 = -1.2(t - 30)$

$d = -1.2t + 66$ ②

Equate ① and ②.

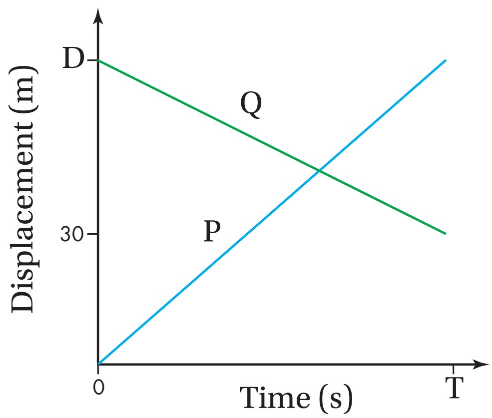
$0.85t - 0.5 = -1.2t + 66$

$2.05t = 66.5$

$t = 32$ (nearest integer)

Time = 5:17 pm

5 a



b For P: $D \div T = 1.25$, so $D = 1.25T$

For Q: $d - 30 = -0.5(t - T)$

When $t = 0$, $d = D$, so $D - 30 = 0.5T$

Substitute $D = 1.25T$ into $D - 30 = 0.5T$

$1.25T - 30 = 0.5T$

$0.75T = 30$

$T = 40$

c $D = 1.25 \times 40 = 50\,\text{m}$

Exercise 15.4B page 414

1 a $-7 \div 20 = -0.35\,\text{m s}^{-2}$

b $10 \times 12 + \frac{1}{2}(12 + 5) \times 20 + 10 \times 5 = 340\,\text{m}$

2 Stage 1 (0 s–6 s): $u = 0$, $v = 10\,\text{m s}^{-1}$, $t = 6\,\text{s}$

Stage 2 (6 s–20 s): steady speed of $10\,\text{m s}^{-1}$ for 14 seconds

Stage 3 (20 s–24 s): $u = 10\,\text{m s}^{-1}$, $v = 25\,\text{m s}^{-1}$, $a = 3.75\,\text{m s}^{-2}$

$v = u + at$; $t = \frac{v-u}{a} = \frac{25-10}{3.75} = 4\,\text{s}$

Stage 4 (24 s–44 s): steady speed of $25\,\text{m s}^{-1}$ for 500 m

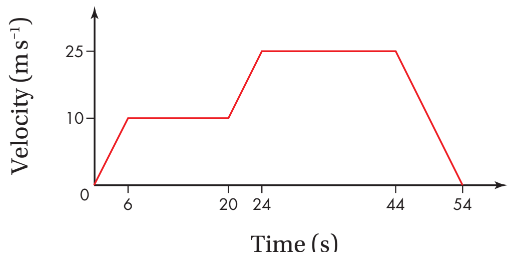
$t = 500 \div 25 = 20\,\text{s}$

Stage 5 (44 s–54 s): $u = 25\,\text{m s}^{-1}$, $v = 0\,\text{m s}^{-1}$, $a = -2.5\,\text{m s}^{-2}$

$v = u + at$; $t = \frac{v-u}{a} = \frac{0-25}{-2.5} = 10\,\text{s}$

a Total time $= 6 + 14 + 4 + 20 + 10 = 54\,\text{s}$

b



c Distance $= \frac{1}{2} \times 6 \times 10 + 14 \times 10 + \frac{1}{2}(10 + 25) \times 4 + 20 \times 25 + \frac{1}{2} \times 10 \times 25$

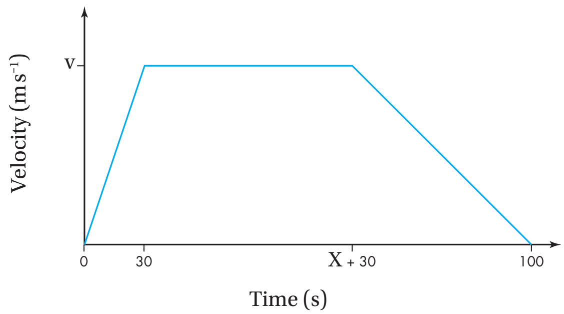
$= 30 + 140 + 70 + 500 + 125 = 865\,\text{m}$

d $a = 10 \div 6 = 1.67\,\text{m s}^{-2}$

3 a Stage 1 (0 s–30 s): $u = 0\,\text{m s}^{-1}$, $v = V\,\text{m s}^{-1}$, $t = 30\,\text{s}$

Stage 2 (30 s–$(X + 30)$ s): steady speed of $V\,\text{m s}^{-1}$ for X seconds

Stage 3 ($(X + 30)$ s–100 s): $u = V\,\text{m s}^{-1}$, $v = 0\,\text{m s}^{-1}$, until 100 s



b $\frac{1}{2} \times 30 \times V = \frac{1}{4}[\frac{1}{2}(X + 100)V]$

$15V = \frac{1}{8}(X + 100)V$

$120 = X + 100$

$X = 20\,s$

c $t = 50\,\text{s}$, $a = -0.9\,\text{m s}^{-2}$, $u = V\,\text{m s}^{-1}$, $v = 0\,\text{m s}^{-1}$

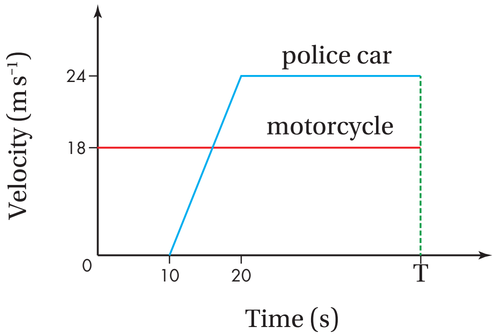
$v = u + at$

$u = v - at$

$V = 0 - (-0.9) \times 50$

$V = 45\,\text{m s}^{-1}$

4 a



b Let T be the time the police car catches up with the motorcyclist.

The distance travelled by both vehicles is the same, so the areas are the same.

$18T = \frac{1}{2}(T - 10 + T - 20) \times 24$

$18T = 12(2T - 30)$

$18T = 24T - 360$

$360 = 6T$

$T = 60\,\text{s}$

c Distance $= 18 \times 60 = 1080\,\text{m}$

5 a First athlete:

Stage 1: $u = 0\,\text{m s}^{-1}$, $t = 10\,\text{s}$, $a = 0.8\,\text{m s}^{-2}$

$v = u + at = 0 + 0.8 \times 10 = 8\,\text{m s}^{-1}$

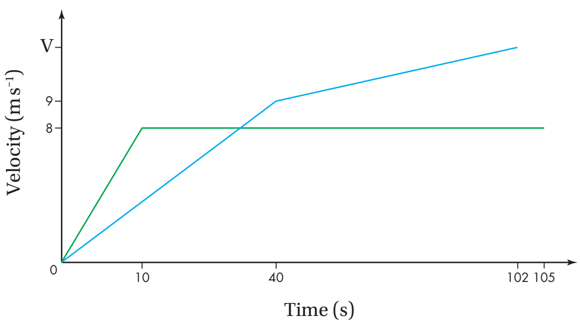
Stage 2: steady speed of $8\,\text{m s}^{-1}$ until 105 s

Second athlete:

Stage 1: $u = 0\,\text{m s}^{-1}$, $t = 40\,\text{s}$, $a = 0.225\,\text{m s}^{-2}$

$v = u + at = 0 + 0.225 \times 40 = 9\,\text{m s}^{-1}$

Stage 2: different acceleration until 102 s



b First athlete: Area $= \frac{1}{2}(105 + 95) \times 8 = 800\,\text{m}$

c Second athlete:

Area $= \frac{1}{2} \times 40 \times 9 + \frac{1}{2}(9 + V) \times 62 = 800$

$180 + 31(9 + V) = 800$

$31V + 279 + 180 = 800$

$31V = 341$

$V = 11\,\text{m s}^{-1}$

6 Stage 1: $u = 0\,\text{m s}^{-1}$, $t = T_1\,\text{s}$, $v = V\,\text{m s}^{-1}$

Substitute into $s = \frac{1}{2}(u + v)t$.

$s = \frac{1}{2}(0 + V) \times T_1 = \frac{1}{2}VT_1$

Stage 2: $u = V\,\text{m s}^{-1}$, $t = T_2\,\text{s}$

There is no acceleration, so substitute into $s = ut$.

$s = VT_2$

Stage 3: $u = V\,\text{m s}^{-1}$, $t = T_3\,\text{s}$, $v = 0\,\text{m s}^{-1}$

Substitute into $s = \frac{1}{2}(u + v)t$.

$s = \frac{1}{2}(V + 0) \times T_3 = \frac{1}{2}VT_3$

The total distance is given by the sum of these distances.

Total distance $= \frac{1}{2}VT_1 + VT_2 + \frac{1}{2}VT_3$

$= \frac{1}{2}V(T_1 + 2T_2 + T_3)$

Since $T = T_1 + T_2 + T_3$, total distance $= \frac{1}{2}V(T + T_2)$, which is the area of the trapezium with parallel sides of T and T_2 and a perpendicular height of V.

7 a Vehicle A:

Stage 1 (0 s–20 s): $u = 0\,\text{m s}^{-1}$, $a = 1.5\,\text{m s}^{-2}$, $t = 20\,\text{s}$

$v = u + at = 0 + 1.5 \times 20 = 30\,\text{m s}^{-1}$

Stage 2 (20 s–90 s): steady speed of $30\,\text{m s}^{-1}$ for 70 s

Stage 3: to rest

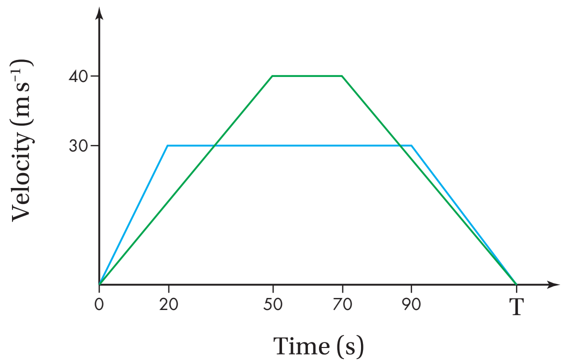
Vehicle B:

Stage 1 (0 s–50 s): $u = 0\,\text{m s}^{-1}$, $a = 0.8\,\text{m s}^{-2}$, $t = 50\,\text{s}$

$v = u + at = 0 + 0.8 \times 50 = 40\,\text{m s}^{-1}$

Stage 2 (50 s–70 s): steady speed of $40\,\text{m s}^{-1}$ for 20 s

Stage 3: to rest



b Same distance for both vehicles.

$\frac{1}{2}(T + 70) \times 30 = \frac{1}{2}(T + 20) \times 40$

$15(T + 70) = 20(T + 20)$

$15T + 1050 = 20T + 400$

$5T = 650$

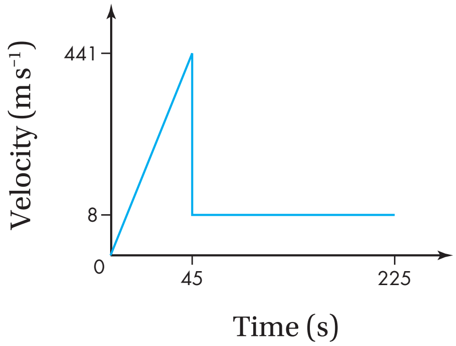
$T = 130\,\text{s}$

c Distance $= \frac{1}{2}(130 + 70) \times 30 = 3000\,\text{m}$

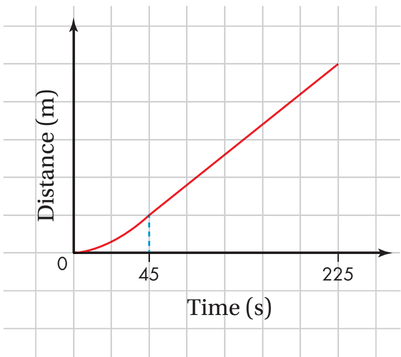
d Vehicle A: $a = -30 \div 40 = -0.75\,\text{m s}^{-2}$

Vehicle B: $a = -40 \div 60 = -0.67\,\text{m s}^{-2}$

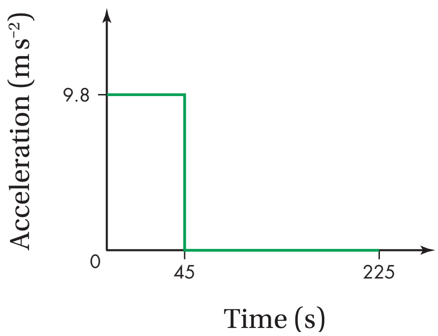
8 a



b



c



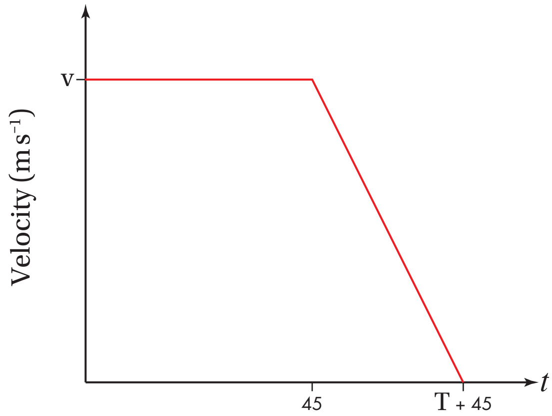
d Distance $= \frac{1}{2} \times 45 \times 441 + 180 \times 8 = 11\,362.5\,\text{m}$

e Shorter. The skydiver will not descend in freefall at g due to air resistance.

9 Stage 1: steady speed of $V\,\text{m s}^{-1}$ for 45 seconds

Stage 2: $u = V\,\text{m s}^{-1}$, $a = -1.6\,\text{m s}^{-2}$, $v = 0\,\text{m s}^{-1}$, $t = T\,\text{s}$

a



b $a = -1.6 = \frac{V}{T}$, so $V = 1.6T$

Area $= \frac{1}{2}(T + 45 + 45) \times V = 2300$

$\frac{1}{2}(T + 90) \times 1.6T = 2300$

$1.6T(T + 90) = 4600$

$1.6T^2 + 144T - 4600 = 0$

$T^2 + 90T - 2875 = 0$

c Completing the square $(T + 45)^2 - 2025 - 2875 = 0$

$(T + 45)^2 = 4900$

$T + 45 = \pm 70$

$T = 25\,\text{s}$ or $-115\,\text{s}$

Hence $T = 25\,\text{s}$

$V = 1.6 \times 25 = 40\,\text{m s}^{-1}$

Exercise 15.5A page 417

1 a $v = \frac{dr}{dt} = 13 - 8t$

$v = 13 - 8 \times 2 = -3\,\text{m s}^{-1}$

b $a = \frac{dv}{dt} = -8\,\text{m s}^{-2}$ (which does not depend on time)

2 a $v = \frac{dr}{dt} = 15t^2 - 14t$

$a = \frac{dv}{dt} = 30t - 14$

b $v = 15(2)^2 - 14(2) = 32\,\text{m s}^{-1}$

c $a = 30(3) - 14 = 76\,\text{m s}^{-2}$

3 a $v = \frac{dr}{dt} = 3t^2 - 16t + 5 = 0$

$(3t - 1)(t - 5) = 0$

$t = \frac{1}{3}\,\text{s}$ or $5\,\text{s}$

b $3t^2 - 16t + 5 > 0$ when $t < \frac{1}{3}$ s and when $t > 5\,\text{s}$

c $a = \frac{dv}{dt} = 6t - 16 > 0$

$t > \frac{8}{3}$ s

d $6t - 16 = 2$

$6t = 18$

$t = 3$

When $t = 3$, $v = 3(3)^2 - 16(3) + 5 = -16\,\text{m s}^{-1}$

When $t = 3$, $r = (3)^3 - 8(3)^2 + 5(3) + 10 = -20\,\text{m}$

4 $18t - t^2 = 65$

$t^2 - 18t + 65 = 0$

$(t - 5)(t - 13) = 0$

$t = 5\,\text{s}$ or $13\,\text{s}$

$a = \frac{dv}{dt} = 18 - 2t$

When $t = 5$, $a = 18 - 2 \times 5 = 8\,\text{m s}^{-2}$

When $t = 13$, $a = 18 - 2 \times 13 = -8\,\text{m s}^{-2}$

5 **a** $r = t^3 - 14t^2 + 48t = 0$

$t(t - 6)(t - 8) = 0$

$t = 6\,\text{s}$ and $8\,\text{s}$

b $v = \frac{dr}{dt} = 3t^2 - 28t + 48$

$a = \frac{dv}{dt} = 6t - 28 = 0$ for minimum velocity

$t = \frac{14}{3}$

$r = (\frac{14}{3})^3 - 14(\frac{14}{3})^2 + 48(\frac{14}{3}) = 20.7\,\text{m}$

6 **a** Substitute $t = 4$: $r = 8\sqrt{4} - \frac{64}{4} = 8 \times 2 - 16 = 0$

b $r = 8t^{\frac{1}{2}} - 64t^{-1}$

$v = \frac{dr}{dt} = 4t^{-\frac{1}{2}} + 64t^{-2}$

$v = 4 \times 4^{-\frac{1}{2}} + 64 \times 4^{-2}$

$v = 4 \times \frac{1}{2} + 64 \times \frac{1}{16}$

$v = 2 + 4 = 6\,\text{m s}^{-1}$

$a = \frac{dv}{dt} = -2t^{-\frac{3}{2}} - 128t^{-3}$

$a = -2 \times 4^{-\frac{3}{2}} - 128 \times 4^{-3}$

$a = -2 \times \frac{1}{8} - 128 \times \frac{1}{64}$

$a = -\frac{1}{4} - 2 = -2\frac{1}{4}\,\text{m s}^{-2}$

7 **a** $4^4 - 8 \times 4^3 + 22 \times 4^2 - 20 \times 4 + q = 33$

$16 + q = 33$

$q = 17$

b $v = \frac{dr}{dt} = 4t^3 - 24t^2 + 44t - 20 = 4$

$4t^3 - 24t^2 + 44t - 24 = 0$

$t^3 - 6t^2 + 11t - 6 = 0$

$(t - 1)(t - 2)(t - 3) = 0$

$t = 1\,\text{s}, 2\,\text{s}$ or $3\,\text{s}$

When $t = 1$, $r = 1^4 - 8 \times 1^3 + 22 \times 1^2 - 20 \times 1 + 17$
$= 12\,\text{m}$

When $t = 2$, $r = 2^4 - 8 \times 2^3 + 22 \times 2^2 - 20 \times 2 + 17$
$= 17\,\text{m}$

When $t = 3$, $r = 3^4 - 8 \times 3^3 + 22 \times 3^2 - 20 \times 3 + 17$
$= 20\,\text{m}$

c $a = \frac{dv}{dt} = 12t^2 - 48t + 44 = 188$

$12t^2 - 48t - 144 = 0$

$t^2 - 4t - 12 = 0$

$(t - 6)(t + 2) = 0$

$t = 6\,\text{s}$ or $-2\,\text{s}$

When $t = 6$, $r = 6^4 - 8 \times 6^3 + 22 \times 6^2 - 20 \times 6 + 17$
$= 257\,\text{m}$

Exercise 15.5B page 420

1 **a** $r = \int v\,dt = \int (t - 2)\,dt = \frac{1}{2}t^2 - 2t + c$

When $t = 6$, $r = 11$

$11 = \frac{1}{2}(6)^2 - 2(6) + c$

$11 = 18 - 12 + c$

$c = 5$

$r = \frac{1}{2}t^2 - 2t + 5$

b When $t = 8$, $r = \frac{1}{2}(8)^2 - 2(8) + 5 = 21\,\text{m}$

c $\frac{1}{2}t^2 - 2t + 5 = 53$

$t^2 - 4t + 10 = 106$

$t^2 - 4t - 96 = 0$

$(t - 12)(t + 8) = 0$

$t = 12\,\text{s}$

$v = 12 - 2 = 10\,\text{m s}^{-1}$

2 **a** $v = \int a\,dt = \int (6t - 20)\,dt = 3t^2 - 20t + c$

When $t = 12$, $v = 223$

$223 = 3(12)^2 - 20(12) + c$

$c = 31$

$v = 3t^2 - 20t + 31$

b $r = \int v\,dt = \int \left(3t^2 - 20t + 31\right) dt$
$= t^3 - 10t^2 + 31t + c_2$

When $t = 7$, $r = 40$

$40 = (7)^3 - 10(7)^2 + 31(7) + c_2$

$c_2 = -30$

$r = t^3 - 10t^2 + 31t - 30$

c $r = (t - 2)(t - 3)(t - 5) = 0$

$t = 2\,\text{s}, 3\,\text{s}$ and $5\,\text{s}$

d $3t^2 - 20t + 31 = 19$

$3t^2 - 20t + 12 = 0$

$(3t - 2)(t - 6) = 0$

$t = \frac{2}{3}\,\text{s}$ and $6\,\text{s}$

3 $v = \int a\,dt = \int (pt - 10)\,dt = \frac{1}{2}pt^2 - 10t + c$

When $t = 2$, $v = 30$

$30 = \frac{1}{2}p(2)^2 - 10(2) + c$

$2p + c = 50$ ①

When $t = 2.5$, $v = 52$

$52 = \frac{1}{2}p(2.5)^2 - 10(2.5) + c$

$25p + 8c = 616$ ②

Multiply ① by 8.

$16p + 8c = 400$ ③

Subtract ③ from ②.

$9p = 216$

$p = 24$

$c = 2$

$v = 12t^2 - 10t + 2$

$r = \int v \, dt = \int (12t^2 - 10t + 2) \, dt = 4t^3 - 5t^2 + 2t + c_2$

When $t = 0$, $r = 0$

$0 = 4(0)^3 - 5(0)^2 + 2(0) + c_2$

$c_2 = 0$

$r = 4t^3 - 5t^2 + 2t$

When $t = 3$, $r = 4(3)^3 - 5(3)^2 + 2(3) = 69\,\text{m}$

4 **a** $6t - 30 = 0$

$t = 5\,\text{s}$

$v = \int a \, dt = \int (6t - 30) \, dt = 3t^2 - 30t + c$

When $t = 0$, $v = 72$

$72 = 3(0)^2 - 30(0) + c$

$c = 72$

$v = 3t^2 - 30t + 72$

When $t = 5$, $v = 3(5)^2 - 30(5) + 72 = -3\,\text{m s}^{-1}$

b $3t^2 - 30t + 72 = 0$

$t^2 - 10t + 24 = 0$

$(t - 4)(t - 6) = 0$

$t = 4\,\text{s}$ and $6\,\text{s}$

c $r = \int v \, dt = \int (3t^2 - 30t + 72) \, dt$

$= t^3 - 15t^2 + 72t + c_2$

When $t = 0$, $r = 0$

$0 = (0)^3 - 15(0)^2 + 72(0) + c_2$

$c_2 = 0$

$r = t^3 - 15t^2 + 72t$

When $t = 4$, $r = 112\,\text{m}$

When $t = 6$, $r = 108\,\text{m}$

When $t = 10$, $r = 220\,\text{m}$

Total distance $= 112 + 4 + 112 = 228\,\text{m}$

5 $v = \int a \, dt = \int (9 - 2t) \, dt = 9t - t^2 + c$

When $t = 0$, $v = 0$

$0 = 9(0) - (0)^2 + c$

$c = 0$

$v = 9t - t^2$

$r = \int v \, dt = \int (9t - t^2) \, dt = \frac{9}{2}t^2 - \frac{1}{3}t^3 + c_2$

When $t = 0$, $r = 0$

$0 = \frac{9}{2}(0)^2 - \frac{1}{3}(0)^3 + c_2$

$c_2 = 0$

$r = \frac{9}{2}t^2 - \frac{1}{3}t^3$

a $r = \frac{9}{2}(2)^2 - \frac{1}{3}(2)^3 = \frac{46}{3}\,\text{m}$

b When $t = 4$, $r = \frac{9}{2}(4)^2 - \frac{1}{3}(4)^3 = \frac{152}{3}\,\text{m}$

When $t = 5$, $r = \frac{9}{2}(5)^2 - \frac{1}{3}(5)^3 = \frac{425}{6}\,\text{m}$

Distance $= \frac{425}{6} - \frac{152}{3} = \frac{121}{6}\,\text{m}$

c When $t = 7$, $r = \frac{9}{2}(7)^2 - \frac{1}{3}(7)^3 = \frac{637}{6}\,\text{m}$

When $t = 9$, $r = \frac{9}{2}(9)^2 - \frac{1}{3}(9)^3 = \frac{243}{2}\,\text{m}$

Distance $= (\frac{243}{2} - \frac{637}{6}) = \frac{46}{3}\,\text{m}$

6 **a** For $0 \leqslant t \leqslant 4$, $v = \int a \, dt = \int \frac{1}{2}(t + 6) \, dt$

$= \int (\frac{1}{2}t + 3) \, dt = \frac{1}{4}t^2 + 3t + c$

When $t = 0$, $v = 2$

$2 = \frac{1}{4}(0)^2 + 3(0) + c$

$c = 2$

$v = \frac{1}{4}t^2 + 3t + 2$

When $t = 4$, $v = \frac{1}{4}(4)^2 + 3(4) + 2 = 18\,\text{m s}^{-1}$

For $4 < t \leqslant 8$, $v = \int a \, dt = \int \frac{320}{t^3} \, dt = \int 320t^{-3} \, dt$

$= -160t^{-2} + c_2$

When $t = 4$, $v = 18$

$18 = -160(4)^{-2} + c_2$

$18 = -10 + c_2$

$c_2 = 28$

$v = 28 - 160t^{-2}$

i When $t = 3$, $v = \frac{1}{4}(3)^2 + 3(3) + 2$

$= 13.25\,\text{m s}^{-1}$

ii When $t = 5$, $v = 28 - 160(5)^{-2} = 21.6\,\text{m s}^{-1}$

b For $0 \leqslant t \leqslant 4$, $r = \int v \, dt = \int (\frac{1}{4}t^2 + 3t + 2) \, dt$

$= \frac{1}{12}t^3 + \frac{3}{2}t^2 + 2t + c_3$

When $t = 0$, $r = 0$

$0 = \frac{1}{12}(0)^3 + \frac{3}{2}(0)^2 + 2(0) + c_3$

$c_3 = 0$

$r = \frac{1}{12}t^3 + \frac{3}{2}t^2 + 2t$

When $t = 4$, $r = \frac{1}{12}(4)^3 + \frac{3}{2}(4)^2 + 2(4) = \frac{112}{3}\,\text{m}$

For $4 < t \leqslant 8$, $r = \int v \, dt = \int (28 - 160t^{-2}) \, dt$

$= 28t + 160t^{-1} + c_4$

When $t = 4$, $r = \frac{112}{3}$

$\frac{112}{3} = 28(4) + 160(4)^{-1} + c_4$

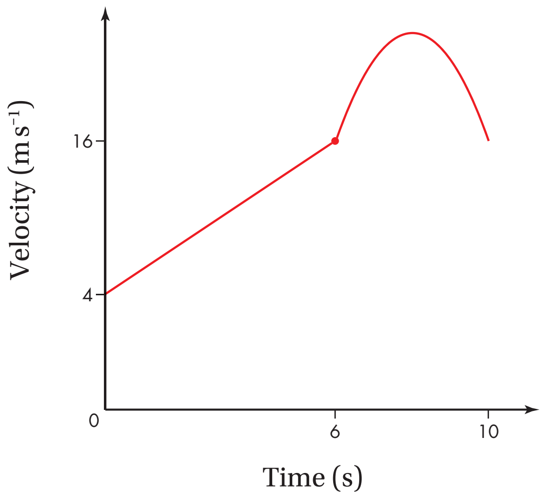
$c_4 = -\frac{344}{3}$

$r = 28t + 160t^{-1} - \frac{344}{3}$

When $t = 8$, $r = 28(8) + 160(8)^{-1} - \frac{344}{3}$

$= \frac{388}{3} = 129.3\,\text{m}$

7 a



b Maximum speed of P is $20\,\text{m s}^{-1}$ when $t = 8$ s.

c i For $0 \leqslant t \leqslant 6$, $a = \frac{dv}{dt} = 2$

When $t = 5$, $a = 2\,\text{m s}^{-2}$

ii For $6 < t \leqslant 10$, $v = 20 - (t - 8)^2$

$= 20 - (t^2 - 16t + 64)$

$= 16t - t^2 - 44$

$a = \frac{dv}{dt} = 16 - 2t$

When $t = 9$, $a = 16 - 2 \times 9 = -2\,\text{m s}^{-2}$

d For $0 \leqslant t \leqslant 6$, $r = \int v\,dt = \int (2t + 4)\,dt$

$= t^2 + 4t + c$

When $t = 0$, $r = 0$

$0 = (0)^2 + 4(0) + c$

$c = 0$

$r = t^2 + 4t$

When $t = 6$, $r = (6)^2 + 4(6) = 60$ m

For $6 < t \leqslant 10$, $r = \int v\,dt = \int \left(16t - t^2 - 44\right) dt$

$= 8t^2 - \frac{1}{3}t^3 - 44t + c_2$

When $t = 6$, $r = 60$

$60 = 8(6)^2 - \frac{1}{3}(6)^3 - 44(6) + c_2$

$c_2 = 108$

$r = 8t^2 - \frac{1}{3}t^3 - 44t + 108$

When $t = 5$, $r = (5)^2 + 4(5) = 45$ m

When $t = 9$, $r = 8(9)^2 - \frac{1}{3}(9)^3 - 44(9) + 108$

$= 117$ m

Alternatively, could use area of trapezium to find distance when $t = 5$:

Area $= \frac{1}{2}(4 + 14) \times 5 = 45$ m

Could also use area of trapezium to find distance when $t = 6$:

Area $= \frac{1}{2}(4 + 16) \times 6 = 60$ m

8 Stage 1: $u = 6\,\text{m s}^{-1}$, $v = 33\,\text{m s}^{-1}$, $t = 18$ s

$s = \left(\frac{u+v}{2}\right)t$

$s = \left(\frac{6+33}{2}\right) \times 18$

$s = 351$ m

Stage 2: For $t > 18$, $a = \frac{1}{4}(t - 12)$

$v = \int a\,dt = \int \frac{1}{4}(t - 12)\,dt = \int \left(\frac{1}{4}t - 3\right) dt$

$= \frac{1}{8}t^2 - 3t + c$

When $t = 18$, $v = 33$

$33 = \frac{1}{8}(18)^2 - 3(18) + c$

$c = \frac{93}{2}$

$v = \frac{1}{8}t^2 - 3t + \frac{93}{2}$

$r = \int v\,dt = \int \left(\frac{1}{8}t^2 - 3t + \frac{93}{2}\right) dt$

$= \frac{1}{24}t^3 - \frac{3}{2}t^2 + \frac{93}{2}t + c_2$

When $t = 18$, $r = 351$

$351 = \frac{1}{24}(18)^3 - \frac{3}{2}(18)^2 + \frac{93}{2}(18) + c_2$

$c_2 = -243$

$r = \frac{1}{24}t^3 - \frac{3}{2}t^2 + \frac{93}{2}t - 243$

When $t = 30$, $r = \frac{1}{24}(30)^3 - \frac{3}{2}(30)^2 + \frac{93}{2}(30) - 243$

$= 927$ m

Exam-style questions 15 page 422

1 a $u = 1.5$, $t = 6$, $s = 90$

$s = ut + \frac{1}{2}at^2$

$90 = 1.5 \times 6 + \frac{1}{2} \times a \times 6^2$

$a = 4.5\,\text{m s}^{-2}$

b $u = 1.5$, $v = 37.5$, $a = 4.5$

$v^2 = u^2 + 2as$

$s = \frac{v^2 - u^2}{2a}$

$s = \frac{37.5^2 - 1.5^2}{2 \times 4.5}$

$s = 156$ m

2 a $a = -9.8$, $v = 0$, $s = 90$

$v^2 = u^2 + 2as$

$u^2 = v^2 - 2as$

$u^2 = 0^2 - 2 \times -9.8 \times 90$

$u^2 = 1764$

$u = 42\,\text{m s}^{-1}$

b $s = 0$, $a = -9.8$, $u = 42$

$s = ut + \frac{1}{2}at^2$

$0 = 42t + \frac{1}{2} \times -9.8 \times t^2$

$4.9t^2 - 42t = 0$

$t(4.9t - 42) = 0$

$t = 42 \div 4.9 = 8.57$ s

3 **a** PQ: $s = 40$, $t = 5$

$s = ut + \frac{1}{2}at^2$

$40 = 5u + 12.5a$

$16 = 2u + 5a$ ①

PR: $s = 60$, $t = 15$

$s = ut + \frac{1}{2}at^2$

$60 = 15u + 112.5a$

$8 = 2u + 15a$ ②

Subtract ① from ②.

$-8 = 10a$

$a = -0.8\,\text{m}\,\text{s}^{-2}$

Substitute for a in ①.

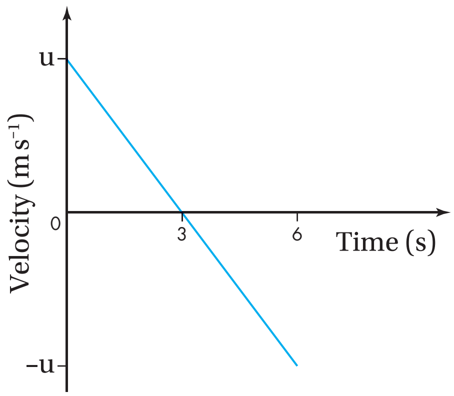
$16 = 2u + 5 \times -0.8$

$20 = 2u$

$u = 10\,\text{m}\,\text{s}^{-1}$

b $a = -0.8\,\text{m}\,\text{s}^{-2}$

4 **a**



b $v = 0$, $s = 44.1$, $t = 3$

$s = \frac{1}{2}(u + v)t$

$44.1 = \frac{1}{2}(u + 0) \times 3$

$u = 29.4\,\text{m}\,\text{s}^{-1}$

c Air resistance

5 For minimum velocity, $a = 0$

$a = 6t - 12 = 0$

$t = 2$

Integrate v to find s.

$s = t^3 - 6t^2 + 11t + c$

$0 = 0 + 0 + 0 + c$

$c = 0$

$s = t^3 - 6t^2 + 11t$

When $t = 2$, $s = (2)^3 - 6(2)^2 + 11(2) = 6\,\text{m}$

6 **a** $u = 18$, $a = 3$, $s = 42$

$v^2 = u^2 + 2as$

$v^2 = 18^2 + 2 \times 3 \times 42$

$v^2 = 576$

$v = 24\,\text{m}\,\text{s}^{-1}$

b For B: $u = 18$, $a = 3$, $s = 42$, $v = 24$

$v = u + at$

$24 = 18 + 3t$

$t = 2$

For A: $u = 36$, $t = 2$, $a = 0$

$s = ut$

$s = 72\,\text{m}$

c B overtakes A when time (T) is same and distance is same.

For B: $s = 18T + 1.5T^2$

For A: $s = 36T$

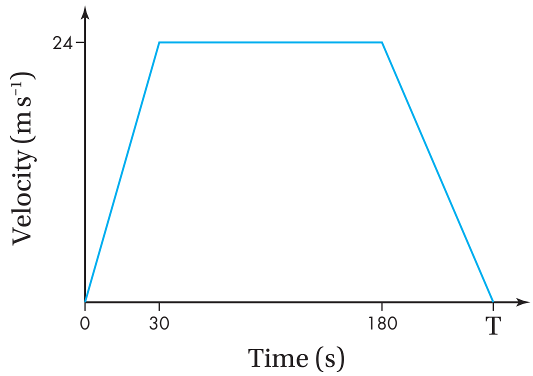
$36T = 18T + 1.5T^2$

$0 = 1.5T^2 - 18T$

$0 = T(1.5T - 18)$

$T = 12\,\text{s}$

7 **a**



b $4500 = \frac{1}{2}(T + 150) \times 24$

$375 = T + 150$

$T = 225\,\text{s}$

c Car: $1920 = \frac{1}{2}(T_1 + T_1 - 30) \times 24$

$T_1 = 95\,\text{s}$

Motorcycle takes $95 - 20 = 75\,\text{s}$.

d Motorcycle: $1920 = \frac{1}{2} \times 75 \times V$

$V = 51.2\,\text{m}\,\text{s}^{-1}$

8 There are four stages to this race.

Stage 1: $u = 0$, $a = 4$, $t = 2$

$s = ut + \frac{1}{2}at^2$

$s = 0 \times 2 + \frac{1}{2} \times 4 \times 2^2$

$s = 8\,\text{m}$

$v = u + at = 0 + 4 \times 2 = 8\,\text{m s}^{-1}$

Stage 2: $u = 8$, $a = \frac{1}{3}$, $t = 3$

$s = ut + \frac{1}{2}at^2$

$s = 8 \times 3 + \frac{1}{2} \times \frac{1}{3} \times 3^2$

$s = \frac{51}{2}\,\text{m}$

$v = u + at = 8 + \frac{1}{3} \times 3 = 9\,\text{m s}^{-1}$

Stage 3: $u = 9$, $a = \frac{3}{13}$, $t = 13$

$s = ut + \frac{1}{2}at^2$

$s = 9 \times 13 + \frac{1}{2} \times \frac{3}{13} \times 13^2$

$s = \frac{273}{2}\,\text{m}$

$v = u + at = 9 + \frac{3}{13} \times 13 = 12\,\text{m s}^{-1}$

Stage 4: $u = 12\,\text{m s}^{-1}$, $a = -2.4\,\text{m s}^{-2}$, $v = 0\,\text{m s}^{-1}$

$s = \frac{v^2 - u^2}{2a} = \frac{0^2 - 12^2}{2 \times -2.4} = 30\,\text{m}$

$t = \frac{v - u}{a} = \frac{0 - 12}{-2.4} = 5\,\text{s}$

Total distance $= 8 + \frac{51}{2} + \frac{273}{2} + 30 = 200\,\text{m}$

Total time $= 2 + 3 + 13 + 5 = 23\,\text{s}$

Average speed $= \frac{\text{total distance}}{\text{total time}} = \frac{200}{23} = 8.70\,\text{m s}^{-1}$

9 a For $0 \leqslant t \leqslant 6$, $v = (8t - t^2)\,\text{m s}^{-1}$

$a = \frac{dv}{dt} = 8 - 2t$

When $t = 5$, $a = 8 - 2 \times 5 = 8 - 10 = -2\,\text{m s}^{-2}$

b For $0 \leqslant t \leqslant 6$, $v = 8t - t^2 = 16 - (t - 4)^2$

The velocity of P is positive for $0 \leqslant t \leqslant 6$.

For $t > 6$, $v = 15 - \frac{1}{2}t$

The particle is at rest when $v = 0$.

$15 - \frac{1}{2}t = 0$

$30 - t = 0$

$t = 30$

The velocity of P is positive for $6 < t < 30$, but negative for $t > 30$.

To find the displacement during the first six seconds, integrate v with respect to t.

$r = \int v\,dt = \int (8t - t^2)\,dt = 4t^2 - \frac{1}{3}t^3 + c$

$0 = 4 \times 0^2 - \frac{1}{3} \times 0^3 + c$

$0 = c$

Hence, $r = 4t^2 - \frac{1}{3}t^3$

When $t = 6$, $r = 4 \times 6^2 - \frac{1}{3} \times 6^3 = 144 - 72 = 72\,\text{m}$

For $t > 6$, $v = 15 - \frac{1}{2}t$

$a = \frac{dv}{dt} = -\frac{1}{2}$

The acceleration is constant.

Since the acceleration is constant, you can use the equations of constant acceleration.

$v = 8t - t^2$

When $t = 6$, $v = 8 \times 6 - 6^2 = 48 - 36 = 12\,\text{m s}^{-1}$

For $6 < t < 30$, $u = 12\,\text{m s}^{-1}$, $v = 0$ and $a = -\frac{1}{2}\,\text{m s}^{-2}$

Use $v^2 = u^2 + 2as$

$s = \frac{v^2 - u^2}{2a}$

$s = \frac{0^2 - 12^2}{2 \times -\frac{1}{2}}$

$s = 144\,\text{m}$

You also need to find the displacement from 30 s to 40 s.

For $t > 30$, $u = 0\,\text{m s}^{-1}$, $t = 10\,\text{s}$ and $a = -\frac{1}{2}\,\text{m s}^{-2}$

Use $s = ut + \frac{1}{2}at^2$

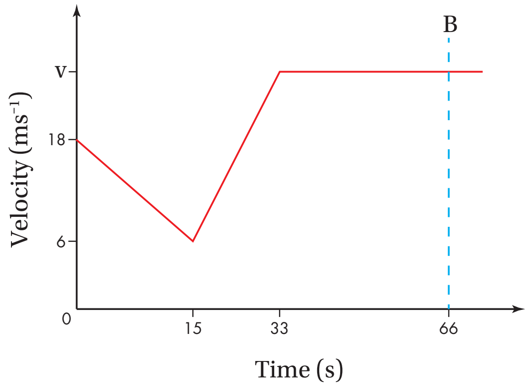
$s = 0 \times 10 + \frac{1}{2} \times -\frac{1}{2} \times 10^2$

$s = -25\,\text{m}$

Total distance $= 72 + 144 + 25 = 241\,\text{m}$

10 a $180 = \frac{1}{2}(6 + 18) \times t$

$t = 15$



b $15 + 18 + 33 = 66\,\text{s}$

c $180 + \frac{1}{2}(6 + V) \times 18 + 33V = 1200$

$180 + 54 + 9V + 33V = 1200$

$42V = 966$

$V = 23\,\text{m s}^{-1}$

11 a $v = \frac{1}{2}t^2 - 6t + c$

$10 = 0 + 0 + c$

$c = 10$

$v = \frac{1}{2}t^2 - 6t + 10$

b $\frac{1}{2}t^2 - 6t + 10 = 0$

$t^2 - 12t + 20 = 0$

$(t - 2)(t - 10) = 0$

$t = 2$ and $10\,\text{s}$

c $s = \frac{1}{6}t^3 - 3t^2 + 10t + c_2$

$s = [\frac{1}{6}(10)^3 - 3(10)^2 + 10(10)] - [\frac{1}{6}(2)^3 - 3(2)^2 + 10(2)]$

$s = \frac{128}{3}\,\text{m} = 42.67\,\text{m}$

12 140 m

13 315 m

14 **a** $r = t^3 + kt^2$

$v = 3t^2 + 2kt$

$3 \times 6^2 + 2k \times 6 = 0$

$k = -9$

$a = 6t + 2k$

$a = 6 \times 13 + 2 \times -9 = 60\text{ m s}^{-2}$

b $r = t^2(t - 9)$

During first 6 s, $r = [6^2(6-9)] - [0] = -108$

From 6 s to 10 s, $r = [10^2(10-9)] - [6^2(6-9)]$
$= 100 - (-108) = 208$

Total distance = 108 + 208 = 316 m

16 Forces

Prior knowledge — page 426

1 $u = 4\text{ m s}^{-1}$, $s = 55\text{ m}$, $t = 5\text{ s}$

a $s = ut + \frac{1}{2}at^2$

$55 = 4 \times 5 + \frac{1}{2} \times a \times 5^2$

$55 = 20 + \frac{25}{2}a$

$35 = \frac{25}{2}a$

$a = 2.8\text{ m s}^{-2}$

b $v = u + at$

$v = 4 + 2.8 \times 5$

$v = 4 + 14$

$v = 18\text{ m s}^{-1}$

2 $|\mathbf{R}| = \sqrt{(-3)^2 + 7^2} = 7.62\text{ N}$

$\tan^{-1}\left(\frac{7}{3}\right) = 66.8°$

Angle with $\mathbf{i} = 180 - 66.8 = 113°$

3 Add ① and ②.

$20 = 4x$

$x = 5$

$y = 10 + 5 = 15$

Exercise 16.1A — page 428

1 **a** $(2\mathbf{i} + 9\mathbf{j}) + (-6\mathbf{i} + 3\mathbf{j}) + (-\mathbf{i} + \mathbf{j}) = (-5\mathbf{i} + 13\mathbf{j})\text{ N}$

b $(6\mathbf{i} - 7\mathbf{j}) + (-8\mathbf{i} - 5\mathbf{j}) + (10\mathbf{i} - 3\mathbf{j}) = (8\mathbf{i} - 15\mathbf{j})\text{ N}$

c $(-0.5\mathbf{i} + 2.1\mathbf{j}) + (2.4\mathbf{i} + 5.6\mathbf{j}) + (\mathbf{i} - 9.5\mathbf{j})\text{ N}$
$= (2.9\mathbf{i} - 1.8\mathbf{j})\text{ N}$

2 **a** $(5\mathbf{i} + \mathbf{j}) + (q\mathbf{i} + 2\mathbf{j}) + (-8\mathbf{i} + p\mathbf{j}) = 0$

$5 + q - 8 = 0$

$q = 3$

$1 + 2 + p = 0$

$p = -3$

b $(p\mathbf{i} - 2q\mathbf{j}) + (p\mathbf{i} - 12\mathbf{j}) + (7\mathbf{i} - q\mathbf{j}) = 0$

$p + p + 7 = 0$

$2p = -7$

$p = -3.5$

$-2q - 12 - q = 0$

$-3q = 12$

$q = -4$

c $(-4\mathbf{i} + 3p\mathbf{j}) + (3q\mathbf{i} + q\mathbf{j}) + (2p\mathbf{i} - 13\mathbf{j}) = 0$

$-4 + 3q + 2p = 0$

$2p + 3q = 4$

$3p + q - 13 = 0$

$3p + q = 13$

Solve simultaneously.

$2p + 3q = 4$

$9p + 3q = 39$

$7p = 35$

$p = 5$

$q = -2$

3 **a** **i** Magnitude $= \sqrt{(-2)^2 + 3^2} = \sqrt{13} = 3.6\text{ N}$

ii $\tan^{-1}\left(\frac{3}{2}\right) = 56.3°$

Angle = 180° − 56.3° = 123.7°

b **i** Magnitude $= \sqrt{8^2 + (-5)^2} = \sqrt{89} = 9.4\text{ N}$

ii $\tan^{-1}\left(\frac{5}{8}\right) = 32.0°$

Angle = −32.0°

c **i** Magnitude $= \sqrt{(-4)^2 + (-4)^2} = \sqrt{32} = 5.7\text{ N}$

ii $\tan^{-1}\left(\frac{4}{4}\right) = 45°$

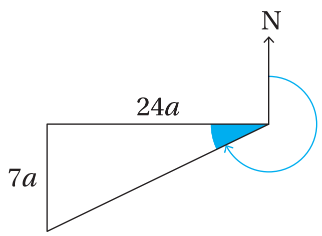
180° − 45° = 135°

Angle = −135°

d **i** Magnitude $= \sqrt{9^2 + 7^2} = \sqrt{130} = 11.4\text{ N}$

ii $\tan^{-1}\left(\frac{7}{9}\right) = 37.9°$

4



a $\tan^{-1}\left(\frac{7}{24}\right) = 16.3°$

270° − 16.3° = 254°

b Magnitude $= \sqrt{(24a)^2 + (7a)^2} = 25a$

$25a = -200$

$a = -8$

5 a $(16\mathbf{i} - 9\mathbf{j}) + a(-2\mathbf{i} + 3\mathbf{j}) = k\mathbf{i}$

$16 - 2a = k$

$-9 + 3a = 0$

$3a = 9$

$a = 3$

$k = 16 - 2 \times 3 = 10$

Magnitude = 10

b $(16\mathbf{i} - 9\mathbf{j}) + a(-2\mathbf{i} + 3\mathbf{j}) = k\mathbf{j}$

$16 - 2a = 0$

$-9 + 3a = k$

$2a = 16$

$a = 8$

$k = -9 + 3 \times 8 = 15$

Magnitude = 15

Exercise 16.2A — page 431

1 a $F = ma$

Write 800 g in SI units.

$F = 0.8 \times 3.1 = 2.48\,\text{N}$

b $F = ma$

$m = \frac{F}{a} = \frac{5200}{4} = 1300\,\text{g}$

2 a $\mathbf{F} = m\mathbf{a}$

$\mathbf{a} = \frac{\mathbf{F}}{m} = \begin{bmatrix} 10 \\ -15 \end{bmatrix} \div 2.5 = \begin{bmatrix} 4 \\ -6 \end{bmatrix}\text{m s}^{-2}$

b $\mathbf{F} = m\mathbf{a}$

$M = (14\mathbf{i} + 35\mathbf{j}) \div (10\mathbf{i} + 25\mathbf{j}) = \frac{7(2\text{i} + 5\text{j})}{5(2\text{i} + 5\text{j})} = 1.4\,\text{kg}$

3 **a** $= \frac{\mathbf{F}}{m} = \begin{bmatrix} -45 \\ 24 \end{bmatrix} \div 1.5 = \begin{bmatrix} -30 \\ 16 \end{bmatrix}\text{m s}^{-2}$

Magnitude of **a** $= \sqrt{(-30)^2 + 16^2} = 34\,\text{m s}^{-2}$

Option **D** ($34\,\text{m s}^{-2}$)

4 a $\mathbf{F} = m\mathbf{a}$

$\mathbf{F} = \frac{2}{3} \times (36\mathbf{i} - 15\mathbf{j}) = (24\mathbf{i} - 10\mathbf{j})\,\text{N}$

$(p\mathbf{i} - 8\mathbf{j}) + (5\mathbf{i} - q\mathbf{j}) + (6\mathbf{i} - 19\mathbf{j}) = (24\mathbf{i} - 10\mathbf{j})$

Equate **i** coefficients.

$p + 5 + 6 = 24$

$p = 13$

Equate **j** coefficients.

$-8 - q - 19 = -10$

$-q = 17$

$q = -17$

b $\sqrt{24^2 + (-10)^2} = 26\,\text{N}$

c $\tan^{-1}\left(\frac{10}{24}\right) = 22.6°$

Angle = −22.6°

d $\mathbf{v} = \mathbf{u} + \mathbf{a}t$

$\mathbf{v} = (4\mathbf{i} + \mathbf{j}) + (36\mathbf{i} - 15\mathbf{j}) \times \frac{1}{3}$

$\mathbf{v} = 4\mathbf{i} + \mathbf{j} + 12\mathbf{i} - 5\mathbf{j}$

$\mathbf{v} = (16\mathbf{i} - 4\mathbf{j})\,\text{m s}^{-1}$

5 a $\mathbf{v} = \mathbf{u} + \mathbf{a}t$

$\mathbf{a} = \frac{\mathbf{v} - \mathbf{u}}{t}$

$\mathbf{a} = \frac{\begin{bmatrix} -1 \\ 7 \end{bmatrix} - \begin{bmatrix} 8 \\ -5 \end{bmatrix}}{6}$

$\mathbf{a} = \begin{bmatrix} -1.5 \\ 2 \end{bmatrix}\text{m s}^{-2}$

b $\mathbf{F} = m\mathbf{a}$

$\mathbf{F} = 0.7 \times \begin{bmatrix} -1.5 \\ 2 \end{bmatrix}$

$\mathbf{F} = \begin{bmatrix} -1.05 \\ 1.4 \end{bmatrix}\text{N}$

c $F = \sqrt{(-1.05)^2 + 1.4^2}$

$F = 1.75\,\text{N}$

6 Liam's solution was correct. The vectors must be added or subtracted before the magnitude is calculated.

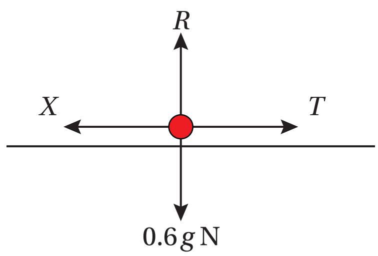
Exercise 16.2B — page 437

1 $W = mg$

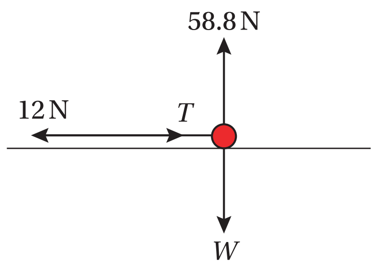
$W = 600 \times 9.8$

$W = 5880\,\text{N}$

2



3 a



b $T - 12 = 0$

$T = 12\,\text{N}$

c $W - 58.8 = 0$

$W = 58.8\,\text{N}$

$W = mg$

$m = \frac{W}{g}$

$m = \frac{58.8}{9.8}$

$m = 6\,\text{kg}$

4 $v = 22\,\text{m s}^{-1}$, $u = 4\,\text{m s}^{-1}$, $t = 15\,\text{s}$

$v = u + at$

$a = \frac{v - u}{t}$

$a = \frac{22 - 4}{15} = 1.2\,\text{m s}^{-2}$

Resultant force, $F = 910 - 190 = 720\,\text{N}$

$F = ma$

$720 = m \times 1.2$

$m = \frac{F}{a} = \frac{720}{1.2} = 600\,\text{kg}$

5 $v = 0\,\text{m s}^{-1}$, $u = 33\,\text{m s}^{-1}$, $s = 240\,\text{m}$

$a = \frac{v^2 - u^2}{2s}$

$a = \frac{0^2 - 33^2}{2 \times 240}$

$a = -2.268\,75\,\text{m s}^{-2}$

Let the braking force $= B\,\text{N}$

Resultant force, $F = (-B - 1100)\,\text{N}$

$F = ma$

$-B - 1100 = 1600 \times -2.26\,875$

$-B - 1100 = -3630$

$B = 2530\,\text{N}$

6 $v = 46\,\text{m s}^{-1}$, $u = 0\,\text{m s}^{-1}$, $s = 460\,\text{m}$

$a = \frac{v^2 - u^2}{2s}$

$a = \frac{46^2 - 0^2}{2 \times 460}$

$a = 2.3\,\text{m s}^{-2}$

Let the resisting force $= D\,\text{N}$

Resultant force, $F = (D - 159)\,\text{N}$

Find the magnitude of the driving force.

$\sqrt{756^2 + 192^2} = 780\,\text{N}$

$F = ma$

$780 - 159 = m \times -2.3$

$m = 270\,\text{kg}$

7 Resultant force, $F = (X - 250)\,\text{N}$

a $F = ma$

$X - 250 = 5 \times 0$

$X = 250\,\text{N}$

b $F = ma$

$X - 250 = 5 \times 2$

$X = 260\,\text{N}$

c $u = 0\,\text{m s}^{-1}$, $s = 2\,\text{m}$, $t = 0.5\,\text{s}$

$s = ut + \frac{1}{2}at^2$

$2 = 0 \times 0.5 + \frac{1}{2} \times a \times 0.5^2$

$2 = 0.125a$

$a = 16\,\text{m s}^{-2}$

$F = ma$

$X - 250 = 5 \times 16$

$X = 330\,\text{N}$

8 Resultant force, $F = 600 - 280 = 320\,\text{N}$

$F = ma$

$320 = 800a$

$a = 320 \div 800 = 0.4\,\text{m s}^{-2}$

$u = 4\,\text{m s}^{-1}$, $s = 385\,\text{m}$, $a = 0.4\,\text{m s}^{-2}$

$v^2 = u^2 + 2as$

$v^2 = 4^2 + 2 \times 0.4 \times 385$

$v^2 = 324$

$v = 18\,\text{m s}^{-1}$

$v = u + at$

$18 = 4 + 0.4t$

$14 = 0.4t$

$t = 35\,\text{s}$

9 $u = 15\,\text{m s}^{-1}$, $v = 30\,\text{m s}^{-1}$, $s = 400\,\text{m}$

$a = \frac{v^2 - u^2}{2s}$

$a = \frac{30^2 - 15^2}{2 \times 400}$

$a = 0.843\,75\,\text{m s}^{-2}$

Let the resistance to motion $= X\,\text{N}$

Resultant force, $F = (700 - X)\,\text{N}$

$F = ma$

$700 - X = 320 \times 0.84375$

$700 - X = 270$

$X = 430\,\text{N}$

Resultant force, $F = (-430 - 500) = -930\,\text{N}$

$F = ma$

$-930 = 320a$

$a = -2.906\,25\,\text{m s}^{-2}$

$u = 30\,\text{m s}^{-1}$, $v = 0\,\text{m s}^{-1}$, $a = -2.906\,25\,\text{m s}^{-2}$

$s = \frac{v^2 - u^2}{2a}$

$s = \frac{0^2 - 30^2}{2 \times -2.90625}$

$s = 155\,\text{m}$

Exercise 16.3A page 440

1 **a** $W = mg$

$W = 8.4 \times 9.8 = 82.32\,\text{N}$

b $W = mg$

$735 = m \times 9.8$

$m = 735 \div 9.8 = 75\,\text{kg}$

$W = mg$

$W = 75 \times 3.75 = 281.25\,\text{N}$

2 $s = 25\,\text{m}$, $u = 0\,\text{m s}^{-1}$

Without air resistance, $a = 9.8\,\text{m s}^{-2}$

$s = ut + \frac{1}{2}at^2$

$25 = 0 \times t + \frac{1}{2} \times 9.8 \times t^2$

$25 = 4.9t^2$

$t = 2.259\,\text{s}$

With air resistance, $0.12g - 0.03 = 0.12a$

$a = 9.55\,\text{m s}^{-2}$

$s = ut + \frac{1}{2}at^2$

$25 = 0 \times t + \frac{1}{2} \times 9.55 \times t^2$

$25 = 4.775t^2$

$t = 2.288\,\text{s}$

Difference $= 0.029\,\text{s}$

3 Resultant force, $F = (T - Mg)\,\text{N}$

$F = ma$

$T - Mg = Ma$

a $40 - 3 \times 9.8 = 3a$

$a = 3.53$

b $8 - M \times 9.8 = M \times 0.6$

$M = 0.769$

c $T - 1.2 \times 9.8 = 1.2 \times 1.5$

$T = 13.56$

4 **a** Resultant force, $F = (T - 45g)\,\text{N}$

$F = ma$

$T - 45g = 45 \times 0.75$

$T = 474.75\,\text{N}$

b Resultant force, $F = (45g - T)\,\text{N}$

$F = ma$

$45g - T = 45 \times 0.75$

$T = 407.25\,\text{N}$

5 Resultant force, $F = (0.35g - 2.8)\,\text{N}$

$F = ma$

$0.35g - 2.8 = 0.35a$

$a = 1.8\,\text{m s}^{-2}$

$u = 3.7\,\text{m s}^{-1}$, $s = 1.9\,\text{m}$, $a = 1.8\,\text{m s}^{-2}$

$v^2 = u^2 + 2as$

$v^2 = 3.7^2 + 2 \times 1.8 \times 1.9$

$v^2 = 20.53$

$v = 4.53\,\text{m s}^{-1}$

$v = u + at$

$4.53 = 3.7 + 1.8t$

$t = 0.462\,\text{s}$

6 Resultant force when lowered, $F = (Mg - 12)\,\text{N}$

$F = ma$

$Mg - 12 = 0.2M$

$M \times 9.8 - 0.2M = 12$

$9.6M = 12$

$M = 1.25$

Let the compression when raised $= C\,\text{N}$

Resultant force when raised, $F = (C - 1.25g)\,\text{N}$

$F = ma$

$C - 1.25g = 1.25 \times 0.2$

$C - 1.25g = 0.25$

$C = 12.5\,\text{N}$

7 Resultant force, $F = (250 - 20g)\,\text{N}$

$F = ma$

$250 - 20g = 20a$

$250 - 200 = 20a$

$a = 2.5\,\text{m s}^{-2}$

$u = 2\,\text{m s}^{-1}$, $v = 5\,\text{m}$, $a = 2.5\,\text{m s}^{-2}$

$v^2 = u^2 + 2as$

$s = \dfrac{v^2 - u^2}{2a}$

$s = \dfrac{5^2 - 2^2}{2 \times 2.5}$

$s = 4.2\,\text{m}$

8 **a** $u = 0\,\text{m s}^{-1}$, $v = 4\,\text{m}$, $s = 2.5\,\text{m}$

$v^2 = u^2 + 2as$

$a = \dfrac{v^2 - u^2}{2s}$

$a = \dfrac{4^2 - 0^2}{2 \times 2.5}$

$a = 3.2\,\text{m s}^{-2}$

To find the mass of the weight:

$W = mg$

$3 = m \times 9.8$

$m = 0.306\,\text{kg}$

Resultant force, $F = (T - 3)\,\text{N}$

$F = ma$

$T - 3 = 0.306 \times 3.2$

$T = 3.98\,\text{N}$

b $u = 4\,\text{m s}^{-1}$, $a = -9.8\,\text{m s}^{-2}$, $v = 0\,\text{m s}^{-1}$

$s = \frac{v^2 - u^2}{2a}$

$s = \frac{0^2 - 4^2}{2 \times -9.8}$

$s = 0.816\,\text{m}$

Maximum height $= 2.5 + 0.816 = 3.32\,\text{m}$

c From maximum height to when it hits the floor, $u = 0\,\text{m s}^{-1}$, $a = 9.8\,\text{m s}^{-2}$, $s = 3.32\,\text{m}$

$v^2 = u^2 + 2as$

$v^2 = 0^2 + 2 \times 9.8 \times 3.32$

$v^2 = 65$

$v = 8.07\,\text{m s}^{-1}$

Exercise 16.4A page 446

1 Resultant force on H $= (3 - T)$ N

Substitute into $F = ma$.

$3 - T = 0.35a$

Resultant force on $G = T$ N

Substitute into $F = ma$.

$T = 0.15a$

Add.

$3 = 0.5a$

$a = 6\,\text{m s}^{-2}$

$T = 0.15a = 0.15 \times 6 = 0.9\,\text{N}$

2 Resultant force on $P = (X - T - 4mg)$ N

Substitute into $F = ma$.

$X - T - 4mg = 4m \times \frac{4}{7}g$ ①

Resultant force on Q $= (T - 3mg)$ N

Substitute into $F = ma$.

$T - 3mg = 3m \times \frac{4}{7}g$ ②

From ②, $T = 3mg + 3m \times \frac{4}{7}g = \frac{33}{7}mg$

Add ① and ②.

$X - 7mg = 7m \times \frac{4}{7}g$

$X = 7mg + 4mg$

$X = 11mg\,\text{N}$

3 Resultant force on trailer $= 500$ N

Substitute into $F = ma$.

$500 = 200a$

$a = 2.5\,\text{m s}^{-2}$

Let the driving force $= D$ N

Resultant force on van $= (D - 500)$ N

Substitute into $F = ma$.

$D - 500 = 700a$

$D - 500 = 700 \times 2.5$

$D = 500 + 1750$

$D = 2250\,\text{N}$

4 Resultant force on car $= (1710 - T - 300k)$ N

Substitute into $F = ma$.

$1710 - T - 300k = 300a$ ①

Resultant force on caravan $= (T - 840k)$ N

Substitute into $F = ma$.

$T - 840k = 840a$ ②

Add ① and ②.

$1710 - 1140k = 1140a$

$1710 = 1140(k + a)$

$k + a = \frac{1710}{1140}$

From ②, $T = 840(k + a)$ N

$T = 840 \times \frac{1710}{1140}$

$T = 1260\,\text{N}$

5 Resultant force on $X = (100 - T - 5g)$ N

Substitute into $F = ma$.

$100 - T - 5g = 5a$

Resultant force on Y $= (T - 3g)$ N

Substitute into $F = ma$.

$T - 3g = 3a$

Add.

$100 - 8g = 8a$

$a = 2.7\,\text{m s}^{-2}$

Whilst connected:

$u = 0\,\text{m s}^{-1}$, $a = 2.7\,\text{m s}^{-2}$, $v = 9\,\text{m s}^{-1}$

$v^2 = u^2 + 2as$

$s = \frac{v^2 - u^2}{2a}$

$s = \frac{9^2 - 0^2}{2 \times 2.7}$

$s = 15\,\text{m}$

When cord snaps:

$u = 9\,\text{m s}^{-1}$, $a = -9.8\,\text{m s}^{-2}$, $s = -15\,\text{m}$

$v^2 = u^2 + 2as$

$v^2 = 9^2 + 2 \times -9.8 \times -15$

$v^2 = 375$

$v = -19.4\,\text{m s}^{-1}$

$v = u + at$

$t = \frac{v-u}{a}$

$t = \frac{-19.4-9}{-9.8}$

$t = 2.89\,\text{s}$

6 $u = 12\,\text{m}\,\text{s}^{-1}$, $v = 32\,\text{m}\,\text{s}^{-1}$, $t = 8\,\text{s}$

$v = u + at$

$a = \frac{v-u}{t}$

$a = \frac{32-12}{8}$

$a = 2.5\,\text{m}\,\text{s}^{-2}$

Resultant force on minibus $= (3000 - T - 475)$
$= (2525 - T)\,\text{N}$

Substitute into $F = ma$.

$2525 - T = 950 \times 2.5 = 2375$

Let mass of cart $= M\,\text{kg}$

Resultant force on cart $= (T - 25)\,\text{N}$

Substitute into $F = ma$.

$T - 25 = 2.5M$

Add.

$2500 = 2.5M + 2375$

$M = 50$

Resultant force on cart when uncoupled $= -25\,\text{N}$

Substitute into $F = ma$.

$-25 = 50a$

$a = -0.5\,\text{m}\,\text{s}^{-2}$

$u = 32\,\text{m}\,\text{s}^{-1}$, $v = 0\,\text{m}\,\text{s}^{-1}$, $a = -0.5\,\text{m}\,\text{s}^{-2}$

$v^2 = u^2 + 2as$

$s = \frac{v^2-u^2}{2a}$

$s = \frac{0^2-32^2}{2\times-0.5}$

$s = 1024\,\text{m}$

7 **a** Let the driving force $= D\,\text{N}$

Resultant force on car $= (D - T - X)\,\text{N}$

Substitute into $F = ma$.

$D - T - X = Ma$

Resultant force on caravan $= (T - 400)\,\text{N}$

Substitute into $F = ma$.

$T - 400 = 750a$

Add.

$D - 400 - X = (M + 750)a$

Substitute $D = 1450\,\text{N}$ and $a = 0.6\,\text{m}\,\text{s}^{-2}$.

$1450 - 400 - X = (M + 750) \times 0.6$

$1050 - X = 0.6M + 450$

$600 = X + 0.6M$ ①

Substitute $D = 2450\,\text{N}$ and $a = 1.4\,\text{m}\,\text{s}^{-2}$.

$2450 - 400 - X = (M + 750) \times 1.4$

$2050 - X = 1.4M + 1050$

$1000 = X + 1.4M$ ②

Subtract ① from ②.

$400 = 0.8M$

$M = 500\,\text{kg}$

b Substitute $M = 500$ into ①.

$600 = X + 0.6 \times 500$

$600 = X + 300$

$X = 300\,\text{N}$

Exercise 16.4B page 448

1 **a** Resultant force $= (R - 55g)\,\text{N}$

$F = ma$

$R - 55g = 55 \times 2.4$

$R = 55g + 55 \times 2.4$

$R = 55(9.8 + 2.4)$

$R = 671\,\text{N}$

b Resultant force $= (55g - R)\,\text{N}$

$F = ma$

$55g - R = 55 \times 1.6$

$R = 55g - 55 \times 1.6$

$R = 55(9.8 - 1.6)$

$R = 451\,\text{N}$

c Resultant force $= (R - 55g)\,\text{N}$

$F = ma$

$R - 55g = 55 \times -0.6$

$R = 55g - 55 \times 0.6$

$R = 55(9.8 - 0.6)$

$R = 506\,\text{N}$

2 **a** For the dog and the floor of the lift, resultant force $= (25g - R)\,\text{N}$

$F = ma$

$25g - R = 25 \times 1.8$

$R = 25g - 25 \times 1.8$

$R = 25(9.8 - 1.8)$

$R = 200\,\text{N}$

b For the lift as a whole, resultant force $= (165g - T)\,\text{N}$

$F = ma$

$165g - T = 165 \times 1.8$

$T = 165g - 165 \times 1.8$

$T = 165(9.8 - 1.8)$

$T = 1320\,\text{N}$

3 Total weight $= 75g + 65g + 550g = 690g\,\text{N}$

Resultant force $= (7480 - 690g)\,\text{N}$

$F = ma$

$7480 - 690g = 690a$

$718 = 680a$

$a = 1.04\,\text{m s}^{-2}$

It is assumed that the lift is stationary when the adults enter.

$u = 0\,\text{m s}^{-1}$, $s = 24\,\text{m}$, $a = 1.04\,\text{m s}^{-2}$

$s = ut + \frac{1}{2}at^2$

$24 = 0 \times t + \frac{1}{2} \times 1.04 \times t^2$

$24 = 0.52t^2$

$t^2 = 46.1$

$t = 6.8\,\text{s}$

4 Total weight with a 160 kg load $= 400g + 160g = 560g\,\text{N}$

Resultant force $= (T - 560g)\,\text{N}$

$F = ma$

$T - 560g = 560 \times 0.75$

$T = 560g + 560 \times 0.75$

$T = 560(9.8 + 0.75)$

$T = 5908\,\text{N}$

Total weight with a 200 kg load $= 400g + 200g = 600g\,\text{N}$

Resultant force $= (5908 - 600g)\,\text{N}$

$F = ma$

$5908 - 600g = 600a$

$28 = 600a$

$a = \frac{7}{150}\,\text{m s}^{-2}$

$u = 0\,\text{m s}^{-1}$, $t = 30\,\text{s}$, $a = \frac{7}{150}\,\text{m s}^{-2}$

$s = ut + \frac{1}{2}at^2$

$s = 0 \times 30 + \frac{1}{2} \times \frac{7}{150} \times 30^2$

$s = 21\,\text{m}$

$v = u + at$

$v = 0 \times \frac{7}{150} \times 30$

$v = 1.4\,\text{m s}^{-1}$

Once the cable snaps, the lift will just be affected by gravity.

$u = 1.4\,\text{m s}^{-1}$, $s = -21\,\text{m}$, $a = -9.8\,\text{m s}^{-2}$

$v^2 = u^2 + 2as$

$v^2 = 1.4^2 + 2 \times -21 \times -9.8$

$v^2 = 413.56$

$v = \sqrt{413.56} = -20.3\,\text{m s}^{-1}$

$-20.3 = 1.4 - 9.8t$

$9.8t = 21.7$

$t = 2.22\,\text{s}$

5 a i and **ii**

Let mass of Margarita $= m\,\text{kg}$

Resultant force on Margarita $= (960 - mg)\,\text{N}$

$F = ma$

$960 - mg = ma$

$960 = mg + ma$ ①

Resultant force on lift $= (4320 - 280g - mg)\,\text{N}$

$F = ma$

$4320 - 280g - mg = (280 + m)a$

$4320 - 280g - mg = 280a + ma$

$4320 - 280g - 280a = mg + ma$ ②

Put ① = ②.

$960 = 4320 - 280g - 280a$

$280a = 616$

$a = 2.2\,\text{m s}^{-2}$

From ①, $m = \frac{960}{g + a}$

$m = \frac{960}{9.8 + 2.2}$

$m = 80\,\text{kg}$

b i, **ii** and **iii**

Let mass of Guillaume $= m_G g\,\text{kg}$ and mass of Josef $= m_J g\,\text{kg}$

Resultant force on Guillaume $= (700 - m_G g)\,\text{N}$

$F = ma$

$700 - m_G g = m_G a$

$700 = m_G g + m_G a$ ①

Resultant force on Josef $= (900 - m_J g)\,\text{N}$

$F = ma$

$900 - m_J g = m_J a$

$900 = m_J g + m_J a$ ②

Resultant force on lift $= (5100 - 280g - m_G g - m_J g)\,\text{N}$

$F = ma$

$5100 - 280g - m_G g - m_J g = (280 + m_G + m_J)a$

$5100 - 280g - m_G g - m_J g = 280a + m_G a + m_J a$

$5100 - 280g - 280a = m_G g + m_G a + m_J g + m_J a$

Substitute from equations ① and ②.

$5100 - 280g - 280a = 700 + 900$

$280a = 756$

$a = 2.7\,\text{m s}^{-2}$

From ①, $m_G = \frac{700}{g+a}$

$m_G = \frac{700}{9.8+2.7}$

$m_G = 56\,\text{kg}$

From ②, $m_J = \frac{900}{g+a}$

$m_J = \frac{900}{9.8+2.7}$

$m_J = 72\,\text{kg}$

Exercise 16.5A — page 453

1 **a** $8g - T = 8a$

$T - 3g = 3a$

$5g = 11a$

$a = 4.45\,\text{m s}^{-2}$

b $T = 3g + 3a$

$T = 3 \times 9.8 + 3 \times 4.45$

$T = 42.8\,\text{N}$

2 **a** Reaction force is equal and opposite to the weight of G.

$R = 3mg$

b **For G**: $T = 3ma$

For H: $2mg - T = 2ma$

Add.

$2mg = 5ma$

$a = \frac{2}{5}g\,\text{m s}^{-2}$

c $T = 3ma$

$T = 3m \times \frac{2}{5}g$

$T = \frac{6}{5}mg\,\text{N}$

3 **a** For the 4 kg particle: $4g - T = 4a$

For the 3 kg particle: $T - 3g = 3a$

Add.

$g = 7a$

$a = \frac{1}{7}g\,\text{m s}^{-2}$

b $T = 3g + 3a$

$T = 3g + 3 \times \frac{1}{7}g$

$T = \frac{24}{7}g\,\text{N}$

c $u = 0\,\text{m s}^{-1}$, $a = \frac{1}{7}g\,\text{m s}^{-2}$, $t = 1.4\,\text{s}$

$s = ut + \frac{1}{2}at^2$

$s = 0 \times 1.4 + \frac{1}{2} \times \frac{1}{7} \times 9.8 \times 1.4^2$

$s = 1.372\,\text{m}$

Since one particle will have travelled 1.372 m upwards and the other 1.372 m downwards, the difference in height is 2.744 m.

4 **a** Let the mass of $P_2 = M\,\text{kg}$

$Mg - T = Ma = M \times \frac{1}{5}g$

$T - 4g = 4a = 4 \times \frac{1}{5}g$

Add.

$(M-4)g = \frac{1}{5}(M+4)g$

$M - 4 = \frac{1}{5}(M+4)$

$5M - 20 = M + 4$

$4M = 24$

$M = 6$

P_2 has a mass of 6 kg.

b The force exerted on the pulley $= 2T$

$T = 4g + 4a = 4g + 4 \times \frac{1}{5}g = 47.04\,\text{N}$

Force exerted on pulley $= 47.04\,\text{N} \times 2 = 94.1\,\text{N}$

5 For A: $T = 3a$

For B: $7.5g - T = 7.5a$

Add.

$7.5g = 10.5a$

$a = 7\,\text{m s}^{-2}$

This acceleration will remain until B hits the ground.

$u = 0\,\text{m s}^{-1}$, $a = 7\,\text{m s}^{-2}$, $s = 2\,\text{m}$

$v^2 = u^2 + 2as$

$v^2 = 0^2 + 2 \times 7 \times 2$

$v^2 = 28$

$v = 2\sqrt{7}\,\text{m s}^{-1}$

$v = u + at$

$t = \frac{v-u}{a}$

$t = \frac{2\sqrt{7}-0}{7}$

$t = 0.756\,\text{s}$

Once B has hit the ground, the string will go slack and A will continue at a constant speed for the remaining 3 m.

$u = 2\sqrt{7}\,\text{m s}^{-1}$, $a = 0\,\text{m s}^{-2}$, $s = 3\,\text{m}$

$s = ut + \frac{1}{2}at^2$

$3 = 2\sqrt{7}t + 0$

$t = 0.567\,\text{s}$

Total time $= 0.756 + 0.567 = 1.32\,\text{s}$

6 Let $x > y$

For the x kg particle: $xg - T = xa$

For the y kg particle: $T - yg = ya$

Add.

$xg - yg = (x+y)a$

Hence $a = \frac{g(x-y)}{x+y}$

Substitute for a in $T = ya + yg$.

$T = y\left(\frac{g(x-y)}{x+y}\right) + yg$

$T = \frac{yg(x-y)+yg(x+y)}{x+y}$

$T = \frac{xyg-gy^2+xyg+gy^2}{x+y}$

$T = \frac{2xyg}{x+y}$ N

This formula is symmetrical in x and y, so the result will also be true if $y > x$.

7 For the ball: $0.5g - T = 0.5a$

For the flowerpot: $T - 2 = 0.3a$

Add.

$0.5g - 2 = 0.8a$

$a = 3.625\text{ m s}^{-2}$

The ball and the flowerpot will both move with an acceleration of 3.625 m s^{-2} until the string snaps, at which point the string will become slack.

$u = 0\text{ m s}^{-1}$, $t = 0.2$ s, $a = 3.625\text{ m s}^{-2}$

$v = u + at$

$v = 0 + 3.625 \times 0.2$

$v = 0.725\text{ m s}^{-1}$

When the string becomes slack, the flowerpot will be slowed down by the 2 N resistive force.

$-2 = 0.3a$

$a = -6\frac{2}{3}\text{ m s}^{-2}$

$u = 0.725\text{ m s}^{-1}$, $v = 0$ m, $a = -6\frac{2}{3}\text{ m s}^{-2}$

$v = u + at$

$t = \frac{v-u}{a}$

$t = \frac{0-0.725}{-6\frac{2}{3}}$

$t = 0.109$ s

Total time $= 0.2 + 0.109 = 0.309$ s

8 Let the mass of P $= m$ kg and let the mass of Q $= (m + 3)$ kg.

Since Q is heavier, Q will fall and P will rise.

For P: $35.91 - mg = ma$

For Q: $(m + 3)g - 35.91 = (m + 3)a$

Add.

$(m + 3)g - mg = (2m + 3)a$

$mg + 3g - mg = (2m + 3)a$

$3g = (2m + 3)a$

$a = \frac{3g}{2m+3}$

Also, $35.91 = mg + ma$

Substitute $a = \frac{3g}{2m+3}$ into $35.91 = mg + ma$

$35.91 = mg + m\frac{3g}{2m+3}$

$35.91(2m + 3) = mg(2m + 3) + 3mg$

$71.82m + 107.73 = 19.6m^2 + 29.4m + 29.4m$

$19.6m^2 - 13.02m - 107.73 = 0$

$m = \frac{13.02 \pm \sqrt{(-13.02)^2 - 4 \times 19.6 \times -107.73}}{2 \times 19.6}$

$m = 2.7$ or -2.04

m cannot be negative, so $m = 2.7$

The mass of P is 2.7 kg and the mass of Q is 5.7 kg.

$a = \frac{3g}{2m+3} = \frac{3g}{2\times2.7+3} = 3.5\text{ m s}^{-2}$

Exam-style questions 16 page 455

1 **a** $\tan^{-1}\left(\frac{48}{14}\right) = 73.7°$

Angle $= 180° - 73.7° = 106°$ (option C)

b $F = ma$

$\mathbf{F} = 0.56 \times (-14\mathbf{i} + 48\mathbf{j})$

$\mathbf{F} = (-7.84\mathbf{i} + 26.88\mathbf{j})$ N

$|\mathbf{F}| = \sqrt{(-7.84)^2 + 26.88^2}$

$|\mathbf{F}| = 28$ N

2 **a** For B: Resultant force $= (3g - T)$ N

$F = ma$

$3g - T = 3a$

For A: Resultant force $= T$ N

$F = ma$

$T = 4a$

Add.

$3g = 7a$

$a = \frac{3}{7}g\text{ m s}^{-2} = 4.2\text{ m s}^{-2}$

$T = 4 \times \frac{3}{7}g$

$T = 16.8$ N

b $u = 0\text{ m s}^{-1}$, $s = 1.3$ m, $a = 4.2\text{ m s}^{-2}$

$s = ut + \frac{1}{2}at^2$

$1.3 = 0 \times t + \frac{1}{2} \times 4.2 \times t^2$

$1.3 = 2.1t^2$

$t^2 = \frac{1.3}{2.1}$

$t = \sqrt{\frac{1.3}{2.1}} = 0.787$ s

3 The student has made the mistake of misreading or assuming that $m > 6$ and therefore that the m kg particle will move downwards and the 6 kg particle upwards. The correct solution is given below.

For the 6 kg particle, resultant force $= (6g - T)$ N

$F = ma$

$6g - T = 6 \times \frac{2}{3}g = 4g$

$T = 2g = 19.6\,\text{N}$

For the m kg particle, resultant force = $(T - mg)$ N

$F = ma$

$T - mg = m \times \frac{2}{3}g$

$2g - mg = \frac{2}{3}mg$

$2g = \frac{5}{3}mg$

$2 = \frac{5}{3}m$

$m = 1.2\,\text{kg}$

4 a Use simultaneous equations.

$-8a - 11b = 10$ ①

$3a - 6b = 3$ ②

Divide ② by 3.

$a - 2b = 1$ ③

Multiply ③ by 8.

$8a - 16b = 8$ ④

Add ① and ④.

$-27b = 18$

$b = -\frac{2}{3}$

$a = 1 + 2b = 1 + 2 \times -\frac{2}{3} = -\frac{1}{3}$

b $-8a - 11b = -2k$ ①

$3a - 6b = 3k$ ②

Divide ② by 3.

$a - 2b = k$ ③

Substitute for k in ①.

$-8a - 11b = -2(a - 2b)$

$-8a - 11b = -2a + 4b$

$-6a = 15b$

$a = -\frac{5}{2}b$

Any values for a and b for which $a = -\frac{5}{2}b$, such as $a = 5$ and $b = -2$.

5 $u = 0\,\text{m s}^{-1}$, $s = 24\,\text{m}$, $t = 4\,\text{s}$

$s = ut + \frac{1}{2}at^2$

$24 = 0 \times 4 + \frac{1}{2} \times a \times 4^2$

$24 = 8a$

$a = 3\,\text{m s}^{-2}$

For A: Resultant force = $(X - 0.5g - T)$ N

$F = ma$

$X - 0.5g - T = 0.5 \times 3 = 1.5$

For B: Resultant force = $(T - 0.2g)$ N

$F = ma$

$T - 0.2g = 0.2 \times 3 = 0.6$

Add.

$X - 0.7g = 2.1$

$X = 0.7g + 2.1$

$X = 8.96\,\text{N}$

6 a When the lift is accelerating, resultant force = $(T - 250g)$ N

$F = ma$

$T - 250g = 250 \times 0.8$

$T = 250(9.8 + 0.8)$

$T = 2650\,\text{N}$

b When accelerating, $u = 0\,\text{m s}^{-1}$, $a = 0.8\,\text{m s}^{-2}$, $t = 15\,\text{s}$

$v = u + at$

$v = 0 + 0.8 \times 15$

$v = 12\,\text{m s}^{-1}$

When decelerating, $u = 12\,\text{m s}^{-1}$, $v = 0\,\text{m s}^{-1}$, $t = 10\,\text{s}$

$v = u + at$

$0 = 12 + a \times 10$

$-10a = 12$

$a = -1.2\,\text{m s}^{-2}$

When the lift is decelerating, resultant force = $(R - 90g)$ N

$F = ma$

$R - 90g = 90 \times -1.2$

$R = 90(g - 1.2)$

$R = 774\,\text{N}$

c The tension is the same for both parts of the cable.

7 a $\mathbf{R} = (q\mathbf{i} - 4q\mathbf{j}) + (7\mathbf{i} + 3\mathbf{j}) + (-2\mathbf{i} - 17\mathbf{j})$
$= (q + 5)\mathbf{i} + (-4q - 14)\mathbf{j} = k(\mathbf{i} - \mathbf{j})$

$q + 5 = k$

$-4q - 14 = -k$

$-4q - 14 = -(q + 5)$

$-4q - 14 = -q - 5$

$-9 = 3q$

$q = -3$

b $k = q + 5 = 2$

$\mathbf{R} = (2\mathbf{i} - 2\mathbf{j})$ N

Magnitude = $\sqrt{2^2 + (-2)^2} = 2.83\,\text{N}$

c $\mathbf{R} = (2\mathbf{i} - 2\mathbf{j}) - (-3\mathbf{i} + 12\mathbf{j}) = (5\mathbf{i} - 14\mathbf{j})$ N

$\tan^{-1}\left(\frac{14}{5}\right) = 70°$

Angle = $-70°$

8 **a** $u = 22\,\text{m s}^{-1}$, $s = -12.5\,\text{m}$, $a = -9.8\,\text{m s}^{-2}$

$v^2 = u^2 + 2as$

$V^2 = 22^2 + 2 \times -9.8 \times -12.5$

$V^2 = 729$

$V = \sqrt{729}$

$V = 27\,\text{m s}^{-1}$

b Resultant force $= (7g - 3000)$ N

$F = ma$

$7g - 3000 = 7a$

$a = -418.77\,\text{m s}^{-2}$

$u = 27\,\text{m s}^{-1}$, $v = 0\,\text{m s}^{-1}$, $a = -418.77\,\text{m s}^{-2}$

$v^2 = u^2 + 2as$

$s = \frac{v^2 - u^2}{2a}$

$s = \frac{0^2 - 27^2}{2 \times -418.77}$

$s = 0.87\,\text{m}$

9 **a** Resultant force for $P_2 = (7 - T - 3) = (4 - T)$ N

$F = ma$

$4 - T = 5.5a$

Resultant force for $P_1 = (T - 2)$ N

$F = ma$

$T - 2 = 4.5a$

Add.

$2 = 10a$

$a = 0.2\,\text{m s}^{-2}$

b $u = 0\,\text{m s}^{-1}$, $a = 0.2\,\text{m s}^{-2}$, $t = 10\,\text{s}$

$v = u + at$

$v = 0 + 0.2 \times 10$

$v = 2\,\text{m s}^{-1}$

c $T = 2 + 4.5a$

$T = 2 + 4.5 \times 0.2$

$T = 2.9\,\text{N}$

d There will now be a thrust force in the rod.

Resultant force for $P_2 = (T - 3)$ N

$F = ma$

$T - 3 = 5.5a$

Resultant force for $P_1 = (-T - 2)$ N

$F = ma$

$-T - 2 = 4.5a$

Add.

$-5 = 10a$

$a = -0.5\,\text{m s}^{-2}$

$u = 2\,\text{m s}^{-1}$, $a = -0.5\,\text{m s}^{-2}$, $v = 0\,\text{m s}^{-1}$

$v^2 = u^2 + 2as$

$s = \frac{v^2 - u^2}{2a}$

$s = \frac{0^2 - 2^2}{2 \times -0.5}$

$s = 4\,\text{m}$

e $T - 3 = 5.5a$

$T = 3 + 5.5 \times -0.5$

$T = 3 - 2.75$

$T = 0.25\,\text{N}$

10 **a** $\tan^{-1}\left(\frac{5}{7}\right) = 35.5°$

$90 + 35.5 = 125.5°$

b $v = u + at$

$7\mathbf{i} - 5\mathbf{j} = (-9\mathbf{i} + 7\mathbf{j}) + 4a$

$16\mathbf{i} - 12\mathbf{j} = 4a$

$\mathbf{a} = (4\mathbf{i} - 3\mathbf{j})\,\text{m s}^{-2}$

$\mathbf{F} = m\mathbf{a}$

$\mathbf{F} = 5 \times (4\mathbf{i} - 3\mathbf{j})$

$\mathbf{F} = (20\mathbf{i} - 15\mathbf{j})$ N

$|\mathbf{F}| = \sqrt{20^2 + (-15)^2}$

$|\mathbf{F}| = 25\,\text{N}$

c $v = u + at$

$v = (-9\mathbf{i} + 7\mathbf{j}) + t(4\mathbf{i} - 3\mathbf{j})$

$v = [(-9 + 4t)\mathbf{i} + (7 - 3t)\mathbf{j}]\,\text{m s}^{-1}$

If v is parallel to $(2\mathbf{i} + \mathbf{j})$, then

$(-9 + 4t)\mathbf{i} + (7 - 3t)\mathbf{j} = k(2\mathbf{i} + \mathbf{j})$

Hence $-9 + 4t = 2k$

$7 - 3t = k$

$-9 + 4t = 2(7 - 3t)$

$-9 + 4t = 14 - 6t$

$10t = 23$

$t = 2.3\,\text{s}$

11 **a** Resultant force for car $= (1500 - T - 300)$

$= (1200 - T)$ N

$F = ma$

$1200 - T = 750a$

Resultant force for trailer $= (T - 100)$ N

$F = ma$

$T - 100 = 250a$

Add.

$1100 = 1000a$

$a = 1.1\,\text{m s}^{-2}$

b $T - 100 = 250a$

$T = 100 + 250 \times 1.1$

$T = 375\,\text{N}$

c Assumed that mass is concentrated at a single point.

d Resultant force for trailer $= (-125 - 100)$
$= -225\,\text{N}$

$F = ma$

$-225 = 250a$

$a = -0.9\,\text{m}\,\text{s}^{-2}$

Resultant force for car $= (-F + 125 - 300)$
$= (-F - 175)\,\text{N}$

$F = ma$

$-F - 175 = 750 \times -0.9$

$-F - 175 = -675$

$F = 500\,\text{N}$

12 a $u = 0\,\text{m}\,\text{s}^{-1}$, $s = 3.75\,\text{m}$, $t = 2.5\,\text{s}$

$s = ut + \frac{1}{2}at^2$

$3.75 = 0 \times 2.5 + \frac{1}{2} \times a \times 2.5^2$

$3.75 = 3.125a$

$a = 1.2\,\text{m}\,\text{s}^{-2}$

b Resultant force for A $= (2.2g - T)\,\text{N}$

$F = ma$

$2.2g - T = 2.2 \times 1.2 = 2.64$

Let mass of B $= m\,\text{kg}$

Resultant force for B $= (T - mg)\,\text{N}$

$F = ma$

$T - mg = 1.2m$

Add.

$2.2g - mg = 1.2m + 2.64$

$2.2g - 2.64 = m(9.8 + 1.2)$

$18.92 = 11m$

$m = 1.72\,\text{kg}$

c Assumed that acceleration is the same for both.

d For B, $u = 0\,\text{m}\,\text{s}^{-1}$, $s = 3.75\,\text{m}$, $a = 1.2\,\text{m}\,\text{s}^{-2}$

$v^2 = u^2 + 2as$

$v^2 = 0^2 + 2 \times 1.2 \times 3.75$

$v^2 = 9$

$v = 3\,\text{m}\,\text{s}^{-1}$

When B returns:

$u = 3\,\text{m}\,\text{s}^{-1}$, $s = 0\,\text{m}$, $a = -9.8\,\text{m}\,\text{s}^{-2}$

$s = ut + \frac{1}{2}at^2$

$0 = 3t + \frac{1}{2} \times -9.8 \times t^2$

$0 = t(3 - 4.9t)$

$t = 0\,\text{s}$ or $\frac{3}{4.9}\,\text{s}$

Hence $t = 0.612\,\text{s}$

13 a Resultant force for truck $= (2500 - 500 - T)$
$= (2000 - T)\,\text{N}$

$F = ma$

$2000 - T = 1600a$

Resultant force for car $= (T - 300)\,\text{N}$

$F = ma$

$T - 300 = 900a$

Add.

$1700 = 2500a$

$a = 0.68\,\text{m}\,\text{s}^{-2}$

b $T = 300 + 900a$

$T = 300 + 900 \times 0.68$

$T = 912\,\text{N}$

c Tensions are the same (equal and opposite).

d Resultant force for truck $= (2500 - 500)$
$= 2000\,\text{N}$

$F = ma$

$2000 = 1600a$

$a = 1.25\,\text{m}\,\text{s}^{-2}$

$u = 25\,\text{m}\,\text{s}^{-1}$, $v = 37\,\text{m}\,\text{s}^{-1}$

When broken, $a = 1.25\,\text{m}\,\text{s}^{-2}$

$$v = u + at$$

$$t = \frac{v - u}{a}$$

$$t = \frac{37 - 25}{1.25} = 9.6\,\text{s}$$

When not broken, $a = 0.68\,\text{m}\,\text{s}^{-2}$

$$v = u + at$$

$$t = \frac{v - u}{a}$$

$$t = \frac{37 - 25}{0.68} = 17.65\,\text{s}$$

Approximately an 8 s difference.

Chapter 1 Algebra and functions 1: Manipulating algebraic expressions

1 **a** $8^{\frac{1}{3}} = +2$

$(+2) \times (+2) \times (+2) = 8$

$(-2) \times (-2) \times (-2) = -8$

b $\frac{1}{4}$

c $\frac{1}{4}y^{-\frac{1}{6}}$

2 **a** $f(x) = x^3 - 6x^2 + 3x + 10$

Use the remainder theorem to find a remainder.

i When $f(x)$ is divided by $(x - 1)$ put $x - 1 = 0$ so $x = 1$, then evaluate $f(1)$ to give the remainder: $f(1) = 1 - 6(1) + 3(1) + 10 = 8$ so the remainder is 8.

ii Put $x + 1 = 0$ so $x = -1$ then evaluate $f(-1)$:

$f(-1) = (-1)^{-3} - 6(-1)^{-2} + 3(-1) + 10$

$= -1 - 6 - 3 + 10$

$= 0$ so the remainder is 0.

b 'Hence' means there is a connection to **part a** of the question.

In **part ii** the remainder is 0 when $f(x)$ is divided by $(x + 1)$ so using the factor theorem we know that $(x + 1)$ is a factor of $f(x)$.

$x^3 - 6x^2 + 3x + 10 = 0$

so $(x + 1)(x - 7x + 10) = 0$

Factorising again:

$(x + 1)(x^2 - 2)(x - 5) = 0$

so $x = -1$, $x = 2$, $x = 5$ are the three solutions of the cubic equation.

3 **a** $(1+x)(2-5x)^3 = (1+x)\left(8-60x+150x^2\right)$

$= 8 - 52x + 90x^2 + \ldots$

b $x = 0.01$

$(1.01)(1.95^3) \approx 7.489$

4 $\frac{a+\sqrt{b}}{c-\sqrt{b}} = \frac{(a+\sqrt{b})}{(c-\sqrt{b})}\frac{\left(c+\sqrt{b}\right)}{\left(c+\sqrt{b}\right)}$

$= \frac{ac+a\sqrt{b}+c\sqrt{b}+b}{c^2-b}$

$= \frac{ac+b}{c^2-b} + \frac{\sqrt{b}(a+c)}{c^2-b}$

5 $\frac{18x^2-71x+28}{4x^3-24x^2+29x+21} = \frac{(2x-7)(9x-4)}{(2x-7)(2x+1)(x-3)}$

$= \frac{9x-4}{(2x+1)(x-3)}$

6 **a** Binomial expansion of $(1 + px)^{-10} = 1 + 10(px) + \ldots$ so the first three terms are:

$1 + 10px + 45px^{-2}$

b Given that the first three terms are 1, $30x$, $27qx^2$:

$10p = 30$ so $p = 3$ and $45p^2 = 27q$ so $45 \times 3 = 27q$ so $q = 5$

c $1.03^{10} = 1.313521899442$

d Expansion only done to term in x^3.

7 **a** $6x^4 - 47x^3 + 81x^2 + 7x - 15$

$= (3x-1)(2x+1)(x-5)(x-3)$

b $x = \frac{1}{3}, -\frac{1}{2}, 5$ or 3

8 **a** $\left(3-\sqrt{x}\right)^4 = 81 - 108x^{\frac{1}{2}} + 54x + x^2 - 12x^{\frac{3}{2}}$

b $\left(\frac{3}{2}-\frac{\sqrt{x}}{2}\right)^4 = \left[\frac{1}{2}\left(3-\sqrt{x}\right)\right]^4 = \left(\frac{1}{2}\right)^4\left(3-\sqrt{x}\right)^{-4}$

$= \frac{1}{16}\left(81-108x^{\frac{1}{2}}+54x+x^2-12x^{\frac{3}{2}}\right)$

9 $\frac{2x^3+x^2-12x+9}{2x^4+x^3-8x^2-x+6} = \frac{(x-1)(2x-3)(x+3)}{(x-1)(x+1)(2x-3)(x+2)}$

$= \frac{x+3}{(x+1)(x+2)}$

10 $\frac{4x}{1-\sqrt{x}} + \frac{3\sqrt{x}+1}{\sqrt{x}} = \frac{4x}{(1-\sqrt{x})}\frac{(1+\sqrt{x})}{(1+\sqrt{x})} + \frac{3\sqrt{x}+1}{\sqrt{x}}\frac{\sqrt{x}}{\sqrt{x}}$

$= \frac{4x+4x\sqrt{x}}{1-x} + \frac{3x+\sqrt{x}}{x}$

$= \frac{x(4x+4x\sqrt{x})}{x(1-x)} + \frac{(1-x)(3x+\sqrt{x})}{x(1-x)}$

$= \frac{4x^2+4x^2\sqrt{x}+3x+\sqrt{x}-3x^2-x\sqrt{x}}{x(1-x)}$

$= \frac{x+3}{1-x} + \frac{\sqrt{x}\left(4x^2-x+1\right)}{x(1-x)}$

Chapter 2 Algebra and functions 2: Equations and inequalities

1 **a** $0 < x < 5$

b **i** From either using the quadratic formula or completing the square, $x = -3 \pm \sqrt{\frac{11}{2}}$

ii $-3 - \sqrt{\frac{11}{2}} < x < -3 + \sqrt{\frac{11}{2}}$

2 Two pairs of solutions, (4, –3) and (–1, 2)

3 **a** $x < -1.07$ or $x > 1.67$

b $-1.07 < x < 1.67$

4 **a** $b < -\sqrt{4ac}$ or $b > \sqrt{4ac}$

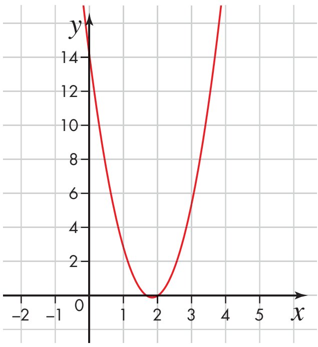
b $b = \pm\sqrt{4ac}$

c $-\sqrt{4ac} < b < \sqrt{4ac}$

5 **a** $(4x-7)(x-2) = 0$ so $x = \frac{7}{4}$ or $x = 2$

b $\left(\frac{15}{8}, \frac{-1}{16}\right)$

c



6 **a** No values for b

b $b = \pm\sqrt{72}$

c $b < -\sqrt{72}$ or $b > \sqrt{72}$

7 **a** $x = \frac{12 \pm \sqrt{14}}{2}, y = \frac{2 \mp \sqrt{14}}{2}$

b As there are two pairs of solutions the linear equation must intersect the circle (twice).

8

$$x^2\sqrt{a} + x\sqrt{b} + \sqrt{c} = 0$$

$$\sqrt{a}\left(x^2 + \frac{\sqrt{b}}{\sqrt{a}}x + \frac{\sqrt{c}}{\sqrt{a}}\right) = 0$$

$$\sqrt{a}\left(\left(x + \frac{\sqrt{b}}{2\sqrt{a}}\right)^2 - \frac{b}{4a} + \frac{\sqrt{c}}{\sqrt{a}}\right) = 0$$

$$\left(x + \frac{\sqrt{b}}{2\sqrt{a}}\right)^2 = \frac{b}{4a} - \frac{4\sqrt{ac}}{4a}$$

$$x + \frac{\sqrt{b}}{2\sqrt{a}} = \frac{\sqrt{b - 4\sqrt{ac}}}{2\sqrt{a}}$$

$$x = -\frac{\sqrt{b}}{2\sqrt{a}} \pm \frac{\sqrt{b - 4\sqrt{ac}}}{2\sqrt{a}}$$

$$= \frac{-\sqrt{b} \pm \sqrt{b - 4\sqrt{ac}}}{2\sqrt{a}}$$

$$= \frac{\sqrt{a}(-\sqrt{b} \pm \sqrt{b - 4\sqrt{ac}})}{2a}$$

9 **a** $x = 2, y = \frac{1}{3}, z = -3$

b Number of equations $\geqslant$ number of variables

10 $25\pi r \leq 450$

$r \leq 5.73$

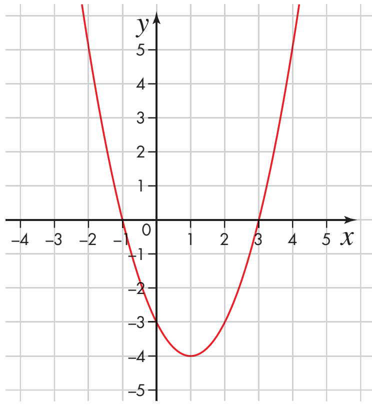
$\pi r^2 > 100$

$r > 5.64$

$5.64 < r \leq 5.73$

Chapter 3 Algebra and Functions 3: Sketching Curves

1 **a**



Points of intersection are $(-1, 0)$, $(3, 0)$ with the x-axis and $(0, -3)$ with the y-axis.

b Points of intersection are where the curve and the line cross.

This will be where the y-coordinate on the curve $y = x^2 - 2x - 3$ (1)

and the y-coordinate on the line $y = x + 2$ (2) are equal.

Solving (1) and (2) as simultaneous equations:

Substitute (2) in (1).

$$x^2 - 2x - 3 = x + 2$$

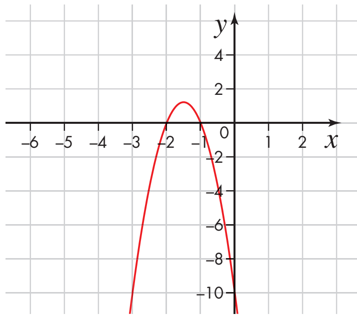
$$x^2 - 3x - 5 = 0$$

Use the quadratic formula with $a = 1$, $b = -3$, $c = -5$.

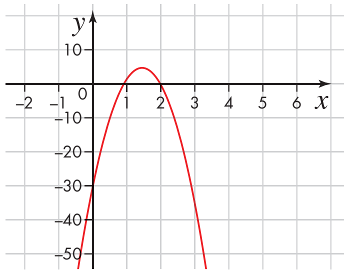
$$x = \frac{3 \pm \sqrt{9 + 20}}{2} = \frac{3 \pm \sqrt{29}}{2}$$

$$y = \frac{7 \pm \sqrt{29}}{2}$$

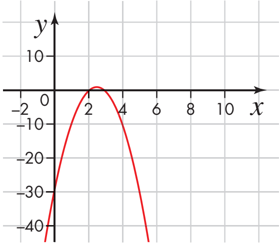
2 **a** **i**



ii



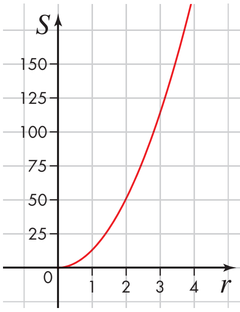
iii



b f$(x+2)$

3 **a** $S = 4\pi r^2$

$S = 196\pi\ \text{cm}^2$



b **i** $S = 64\pi\ \text{cm}^2$

ii $r = +\sqrt{72}$ cm

4 $x \propto y^2$ and $z \propto \sqrt[3]{x}$

$x = ky^2$

$z = K\sqrt[3]{x}$

Substituting gives:

$z = K\sqrt[3]{ky^2}$

$z \propto \sqrt[3]{y^2}$ or $z \propto y^{\frac{2}{3}}$

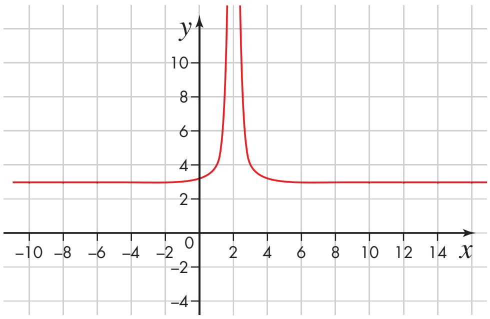
b $z = \dfrac{21(5)^{\frac{2}{3}}}{4}$

5 **a** f$(x-2)$

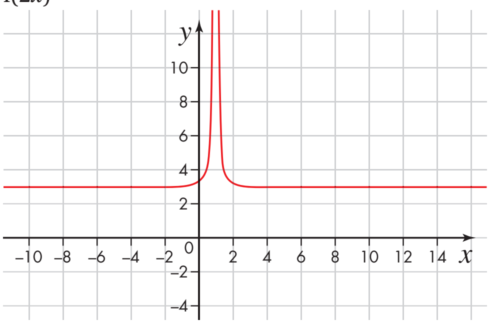
b f$(x)+5$

c $-$f(x)

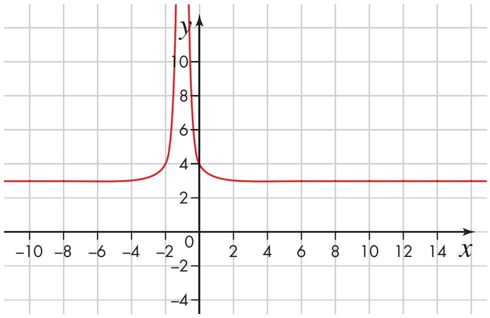
6 **a**



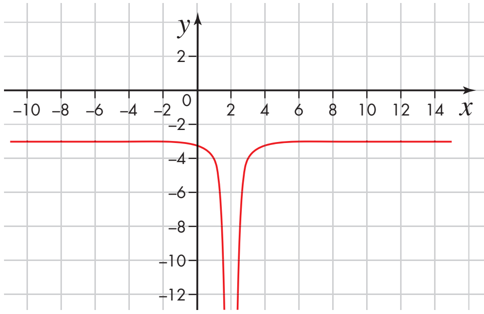
b **i** f$(2x)$



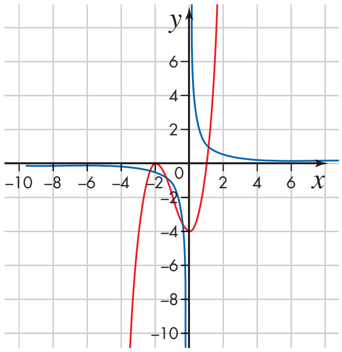
ii f$(x+3)$



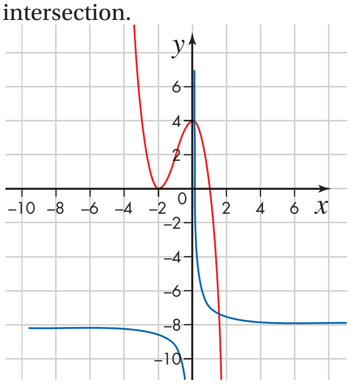
iii $-$f(x)



7 **a** Asymptotes at $x = 0$ and $y = 0$. Four points of intersection.

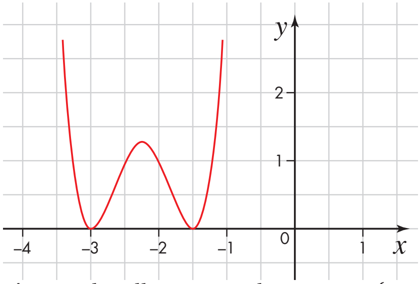


b Asymptotes at $x = 0$ and $y = -8$. Two points of intersection.



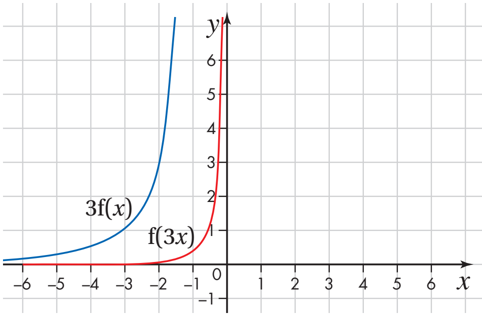
8 **a** $(x+1)^3(2x-1)$

b



The graph will intercept the y-axis at (0, 81)

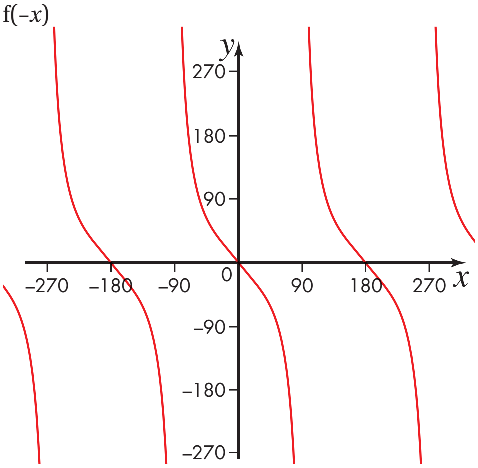
9 **a**



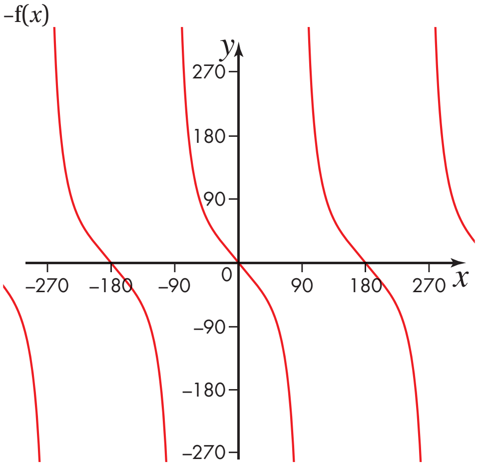
b $f(3x+2)$ and $f(x+3)$

c $f(x)=3^{3x+2}$ and $f(x)=3^{x+3}$

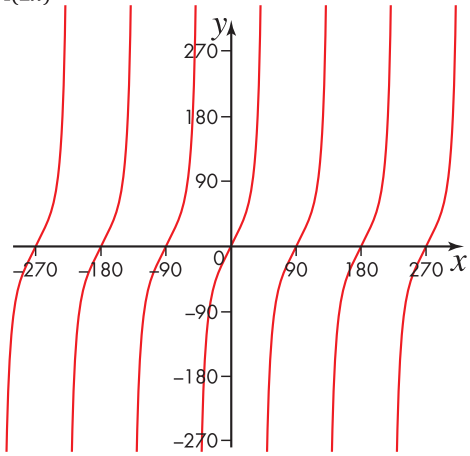
10 **a** $f(-x)$



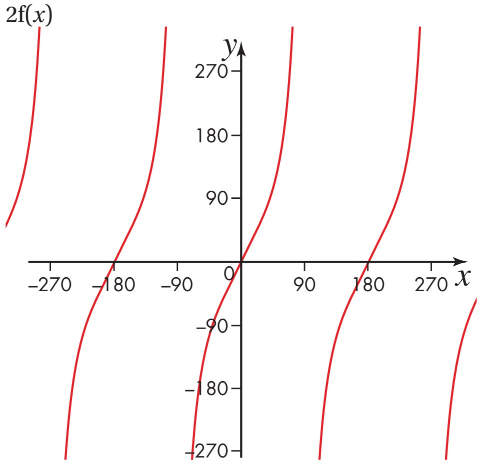
b $-f(x)$



c $f(2x)$



d $2f(x)$



e $f(-x) = -f(x)$ but $f(2x) \neq 2f(x)$

Chapter 4 Coordinate geometry 1: Equations of straight lines

1 **a** Rearrange to find y in terms of x:

$$2y-3x-4=0$$
$$2y=3x+4$$
$$y=\frac{3}{2}x+2$$

Gradient is $\frac{3}{2}$.

b Use $y-y_1=m(x-x_1)$:

$$y-2=\frac{3}{2}(x-4)$$
$$2y-4=3x-12$$
$$2y-3x+8=0$$
$$-3x+2y+8=0$$

2 Use $\frac{y-y_1}{y_2-y_1}=\frac{x-x_1}{x_2-x_1}$:

$$\frac{y-1}{2-1}=\frac{x-2}{-4-2}$$
$$\frac{y-1}{1}=\frac{x-2}{-6}$$
$$6-6y=x-2$$
$$-6y-x+8=0$$
$$x+6y-8=0$$

Remember to write your answer in the form $ax+by+c=0$, which means writing all of the equation on the left-hand side of the '=', starting with the coefficient of *x*.

3 **a** $y=1.25x$

b The rate at which the sunflower grows in cm per day, which is 1.25 cm per day.

c $3.1 \times 100 = 1.25x$, $x = 310 \div 1.25 = 248$ days

d 248 days is about 8 months. Although this is possible, a sunflower is unlikely to grow at a steady rate for this long, so a linear relationship is not valid.

4 **a** $y-3x-2=0$

$$y=3x+2$$

Gradient is 3.

b Gradient of perpendicular so use $mm'=-1$:

Perpendicular gradient $=-\frac{1}{3}$ as $3\times\frac{-1}{3}=-1$

Use $y-y_1=m(x-x_1)$:

$$y-4=-\frac{1}{3}(x-3)$$
$$3y-12=-x+3$$
$$3y+x-15=0$$
$$x+3y-15=0$$

5 **a** $2y+x-4=0$

$$y=-\frac{1}{2}x+2$$

Gradient is $\frac{-1}{2}$

$y-2x+1=0$

$$y=2x-1$$

Gradient is 2.

Product of gradients is $-\frac{1}{2}\times 2=-1$

Therefore the lines are perpendicular.

b $2y+x-4=0$ ①

$y-2x+1=0$ ②

Multiply ① by 2:

$4y+2x-8=0$ ③

$y-2x+1=0$ ②

Add ② and ③.

$5y-7=0$, $y=1.4$

Substitute $y = 1.4$ into ①.

$$2y+x-4=0$$
$$2.8+x-4=0$$
$$x=1.2$$

Coordinates of the point of intersection are (1.2, 1.4).

6 Find the gradients of the lines joining the points. If the triangle is right angled then two of the lines will be perpendicular to each other.

Use $m=\frac{y_2-y_1}{x_2-x_1}$

Line joining (–3, 3) and (4, 4): gradient $=\frac{4-3}{4-(-3)}=\frac{1}{7}$

Line joining (4, 4) and (3, 6): gradient $=\frac{6-4}{3-4}=\frac{2}{-1}=-2$

Line joining (–3, 3) and (3, 6): gradient $=\frac{6-3}{3-(-3)}=\frac{3}{6}$

$\frac{3}{6}\times(-2)=\frac{-6}{6}=-1$ so two of the sides are perpendicular. So the triangle is right angled.

7 **a** $y=18x+15$

b Plumber A: $15\times 3.5+30=£82.50$

Plumber B: $18\times 3.5+15=£78$

Plumber B is cheaper.

c $18x+15=15x+30$

$$3x=15$$
$$x=5$$

5 hours

8 Use $m=\frac{y_2-y_1}{x_2-x_1}$:

Gradient of line joining (–3, –2) and (1, –1)

$$=\frac{\frac{1}{4}-1-(-2)}{1-(-3)}=\frac{1}{4}$$

Gradient of line joining (–3, –2) and (6, $\frac{1}{4}$)

$$=\frac{-(-2)}{6-(-3)}=\frac{2\frac{1}{4}}{9}=\frac{\frac{9}{4}}{9}=\frac{1}{4}$$

As the two lines have a point in common and have the same gradient, they must be in a straight line.

9 $9y+4x-18=0$

Rearrange to find the gradient: $9y=-4x+18$

$$y=-\frac{4}{9}x+2$$

So, gradient $=-\frac{4}{9}$.

Gradient of perpendicular so use $mm'=-1$.

$-\frac{4}{9}\times\frac{9}{4}=-1$

Gradient of perpendicular $=\frac{9}{4}$

Now use $y-y_1=m(x-x_1)$:

$$y-6=\frac{9}{4}(x-3)$$
$$4y-24=9x-27$$
$$4y-9x+3=0$$

$9x-4y-3=0$ or $-9x+4y+3=0$

10 The points A and B have coordinates (1, 2) and (3, 8), respectively.

The perpendicular bisector passes through the midpoint of AB, (2, 5).

Gradient of AB $=\frac{8-2}{3-1}=\frac{6}{2}=3$

Gradient of perpendicular so use $mm'=-1$:

$-\frac{1}{3}\times 3=-1$

Gradient of perpendicular bisector has gradient $-\frac{1}{3}$ and passes through (2, 6).

Now use $y-y_1=m(x-x_1)$:

$$y-5=-\frac{1}{3}(x-2)$$
$$3y-15=-x+2$$
$$3y=-x+17$$
$$y=-\frac{1}{3}x+\frac{17}{3}$$

11 As the point lies on the line, the coordinates $(4k, 4k+3)$ must satisfy $y=\frac{4}{3}x-1$.

When $x=4k$, $y=4k+3$

$$4k+3=\frac{4}{3}(4k)-1$$
$$12k+9=16k-3$$
$$12=4k$$
$$k=3$$

12 a The gradient m gives the rate at which the depth changes in cm per minute. In tank B this is 4 cm per minute and in tank A this is 3 cm per minute. So the depth of water is increasing at a faster rate in tank B.

The constant term c gives the depth of water when x is 0, i.e. it gives the depth of water in the tank when the filling started. Tank B had 10 cm of water in it at the start and tank A had 20 cm of water in it at the start.

b When the depth of water is the same in both tanks $y=3x+20$ and $y=4x+10$.

$$3x+20=4x+10$$
$$10=x$$

When the tanks have been filling for 10 minutes the depth of water is the same in both tanks.

c Note: the tanks' heights are in metres (y is in cm) and the tanks do not have the same height.

Tank A: $200=3x+20$

$180=3x$

$x=60$

Tank A is full after 60 minutes.

Tank B: $260=4x+10$

$250=4x$

$x=62.5$

Tank B is full after 62.5 minutes.

Tank A is filled first.

13 a P(p, p – 3) lies on the line so the coordinates must satisfy $2x-5y+3=0$.

Substitute $x=p$ and $y=p-3$.

$$2p-5(p-3)+3=0$$
$$2p-5p+15+3=0$$
$$-3p+18=0$$
$$3p=18$$
$$p=6$$

b $2x-5y+3=0$

$$5y=2x+3$$
$$y=\frac{2}{5}x+\frac{3}{5}$$

Gradient is $\frac{2}{5}$.

c Gradient of perpendicular so use $mm'=-1$:

$\frac{-5}{2}\times\frac{2}{5}=-1$

Gradient of perpendicular is $\frac{-5}{2}$.

Now use $y-y_1=m(x-x_1)$:

$$y-1=-\frac{5}{2}(x-1)$$
$$2y-2=-5x+5$$
$$2y+5x-7=0$$
$$5x+2y-7=0$$

d Make sure you use the correct equations here.

$2x-5y+3=0$

$2x-5y=-3$ ①

$3x-4y=6$ ②

Multiply ① by 3 and ② by 2.

$6x-15y=-9$ ③

$6x-8y=12$ ④

Subtract ③ from ④.

$$7y=21$$
$$y=3$$

Substitute this value in ①.

$$2x-15=-3$$
$$2x=12$$
$$x=6$$

R has coordinates (6, 3).

14 To find the coordinates of D the equations of AD and DC are needed.

AB has equation $3y-2x-5=0$ so $3y=2x+5$.

$y=\frac{2}{3}x+\frac{5}{3}$

AB has gradient $\frac{2}{3}$.

For AD:

AD is perpendicular to AB, so gradient of AD is $-\frac{3}{2}$.

A has coordinates (–1, 1).

Now use $y-y_1=m(x-x_1)$:

$$y-1=-\frac{3}{2}(x--1)$$
$$2y-2=-3x-3$$
$$2y=-3x-1$$

For DC:

DC is parallel to AB and passes through C (8.5, 3).

Use $y-y_1=m(x-x_1)$:

$$y-3=\frac{2}{3}(x-8.5)$$
$$3y-9=2x-17$$
$$3y=2x-8$$

Now solve the equations simultaneously for D:

$$2y=-3x-1 \quad ①$$
$$3y=2x-8 \quad ②$$

Multiply ① by 3 and ② by 2.

$$6y=-9x-3 \quad ③$$
$$6y=4x-16 \quad ④$$
$$(6y=)-9x-3=4x-16$$
$$13=13x$$
$$x=1$$

Substitute into ②. $3y=2-8$

$$3y=-6$$
$$y=-2$$

Coordinates of D are (1, –2).

15 Gradient of AD is –2.

Gradient of CD and BA is $\frac{1}{2}$.

Find the equation of AB, with gradient $\frac{1}{2}$ through (4, 5).

Use $y-y_1=m(x-x_1)$:

$$y-5=\frac{1}{2}(x-4)$$
$$2y-10=x-4$$
$$2y=x+6$$

AB meets AD at A:

$$2y=x+6 \quad ①$$
$$y=-2x+3 \quad ②$$

Multiply ② by 2.

$$2y=-4x+6 \quad ③$$
$$(2y=)-4x+6=x+6$$
$$0=5x$$
$$x=0,\text{ so } y=3$$

Find the equation of CD, with gradient $\frac{1}{2}$ through (2, 9).

Use $y-y_1=m(x-x_1)$:

$$y-9=\frac{1}{2}(x-2)$$
$$2y-18=x-2$$
$$2y=x+16$$

CD meets AD at D:

$$y=-2x+3 \quad ①$$
$$2y=x+16 \quad ②$$

Multiply ① by 2.

$$2y=-4x+6 \quad ③$$
$$(2y=)x+16=-4x+6$$
$$5x=-10$$
$$x=-2\text{, so } y=7$$

Coordinates are (0, 3) and D (–2, 7).

Chapter 5 Coordinate geometry 2: Circles

1 **a** $(x-4)^2-16+(y+10)^2-100+35=0$

$(x-4)^2+(y+10)^2=81$

Centre is (4, –10)

b Radius = $\sqrt{81}=9$

2 **a** $\sqrt{(3-(-9))^2+((-8)-(-3))^2}=\sqrt{12^2+(-5)^2}=13$

b Gradient of radius = $\frac{(-3)-(-8)}{(-9)-3}=-\frac{5}{12}$

Gradient of tangent = $\frac{12}{5}$

$$y-(-3)=\frac{12}{5}(x-(-9))$$
$$5(y+3)=12(x+9)$$
$$5y+15=12x+108$$
$$12x-5y+93=0$$

3 **a** Gradient of $L_1=\frac{(-3)-(-5)}{7-9}=-1$

$$y-(-5)=-1(x-(9))$$
$$y+5=-x+9$$
$$x+y=4$$

b Substitute $y = 4 - x$ into the equation of C.

$(x + 3)^2 + (4 - x - 2)^2 = 17$
$(x + 3)^2 + (2 - x)^2 = 17$
$x^2 + 6x + 9 + 4 - 4x + x^2 = 17$
$2x^2 + 2x - 4 = 0$
$x^2 + x - 2 = 0$
$(x + 2)(x - 1) = 0$
$x = -2$ or 1
$y = 6$ or 3

Midpoint $= \left(\dfrac{-2+1}{2}, \dfrac{6+3}{2}\right) = (-\tfrac{1}{2}, \tfrac{9}{2})$

4 a $\sqrt{(5-(-3))^2 + (10-4)^2} = \sqrt{8^2 + 6^2} = 10$ m

b Centre of circle $= \left(\dfrac{-3+5}{2}, \dfrac{4+10}{2}\right) = (1, 7)$

Radius of circle = 5

$(x - 1)^2 + (y - 7)^2 = 25$
$x^2 - 2x + 1 + y^2 - 14y + 49 = 25$
$x^2 + y^2 - 2x - 14y + 25 = 0$

c Gradient of radius $= \dfrac{10-7}{5-1} = \dfrac{3}{4}$

Gradient of tangent $= -\dfrac{4}{3}$

$y - 10 = -\dfrac{4}{3}(x - 5)$
$3(y - 10) = -4(x - 5)$
$3y - 30 = -4x + 20$
$4x + 3y = 50$

5 a Equation of circle is: $(x - 5)^2 + (y - 3)^2 = 5^2$

When $y = 0$, $(x - 5)^2 + (-3)^2 = 25$
$(x - 5)^2 + 9 = 25$
$(x - 5)^2 = 16$
$x - 5 = \pm 4$
$x = 1$ or 9

A(1, 0), B(9, 0)

b AB = 8

Area $= \tfrac{1}{2} \times 8 \times 3 = 12$

6 a Radius $= \sqrt{(5-(-7))^2 + ((-2)-4)^2}$
$= \sqrt{12^2 + (-6)^2} = 180$

$(x + 2)^2 + (y - 5)^2 = 180$

b Let M be the point where the perpendicular bisector XY meets the line segment AO.

$OX = \sqrt{180} = 6\sqrt{5}$
$OM = 3\sqrt{5}$
$XY = 2 \times \sqrt{(6\sqrt{5})^2 - (3\sqrt{5})^2} = 6\sqrt{15}$

7 a Midpoint of YZ = (5, –2)

Gradient of YZ $= \tfrac{1}{2}$

Gradient of bisector = –2

Equation of bisector is $y - (-2) = -2(x - 5)$
$y + 2 = -2x + 10$
$2x + y = 8$

b Midpoint of XZ $= (\tfrac{7}{2}, \tfrac{7}{2})$

Gradient of XZ $= -\tfrac{1}{3}$

Gradient of bisector = 3

Equation of bisector is $y - \tfrac{7}{2} = 3(x - \tfrac{7}{2})$
$2y - 7 = 6x - 21$
$2y = 6x - 14$
$y = 3x - 7$

Substitute in $2x + y = 8$.

$2x + (3x - 7) = 8$
$5x = 15$
$x = 3$
$y = 2$

Centre = (3, 2)

c Radius $= \sqrt{(11-3)^2 + (1-2)^2} = \sqrt{8^2 + (-1)^2} = 65$

$(x - 3)^2 + (y - 2)^2 = 65$

8 a $(x + 4)^2 + (3x + 1 - 9)^2 = 40$
$(x + 4)^2 + (3x - 8)^2 = 40$
$x^2 + 8x + 16 + 9x^2 - 48x + 64 = 40$
$10x^2 - 40x + 40 = 0$
$x^2 - 4x + 4 = 0$
$(x - 2)^2 = 0$

Repeated root, so line is a tangent.

b Centre is the midpoint of the two points on the circumference where the tangent is of the form $y = 3x + c$.

When $x = 2$, $y = 3(2) + 1 = 7$
$x = 2 + 2((-4) - 2) = -10$
$y = 7 + 2(9 - 7) = 11$
$y - 11 = 3(x - (-10))$
$y = 3x + 41$

9 a Centre (–2, 11), radius $= \sqrt{49} = 7$

b $PX = \sqrt{((-2)-(-3))^2 + (11-24)^2}$
$= \sqrt{1^2 + (-13)^2} = \sqrt{170}$

$PX^2 = 170$
$TX^2 = 49$
$PT^2 = PX^2 - TX^2 = 170 - 49 = 121$
$PT = \sqrt{121} = 11$

10 Centre of C = (7, 13)

Since gradient of tangent $= \tfrac{1}{2}$, gradient of radius = –2

Equation of diameter is: $y - 13 = -2(x - 7)$
$y = -2x + 27$

Substitute $y = -2x + 27$ into $x^2 + y^2 - 14x - 26y + 138 = 0$.

$x^2 + (-2x + 27)^2 - 14x - 26(-2x + 27) + 138 = 0$
$x^2 + 4x^2 - 108x + 729 - 14x + 52x - 702 + 138 = 0$
$5x^2 - 70x + 165 = 0$
$x^2 - 14x + 33 = 0$
$(x - 3)(x - 11) = 0$
$x = 3$ or 11

When $x = 3$, $y = -2(3) + 27 = 21$

When $x = 11$, $y = -2(11) + 27 = 5$

Tangent at point (3, 21): $y - 21 = \frac{1}{2}(x - 3)$

$y = \frac{1}{2}x + \frac{39}{2}$

Tangent at point (11, 5): $y - 5 = \frac{1}{2}(x - 11)$

$y = \frac{1}{2}x - \frac{1}{2}$

Chapter 6 Trigonometry

1 $\tan x° = \pm\sqrt{2}$

If $\tan x° = \sqrt{2}$ then $x = 54.7$ or 234.7

If $\tan x° = -\sqrt{2}$ then $x = 125.3$ or 305.3

2 First find angle C: $\frac{\sin C}{15} = \frac{\sin 85°}{20}$; $\sin C = 0.7471$ and so $C = 48.3°$

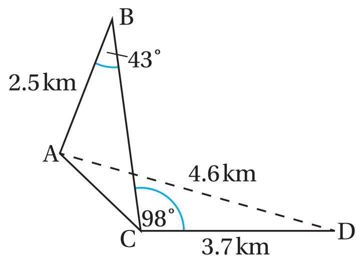
Then $B = 180 - (85 + 48.3) = 46.7°$

Area $= \frac{1}{2} \times 15 \times 20 \times \sin 46.7° = 109 \text{ m}^2$

3 **a** $\cos^2\theta = 1 - \frac{1}{3} = \frac{2}{3}$ so $\cos\theta = -\sqrt{\frac{2}{3}}$ (taking the negative square root).

b $\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{1}{\sqrt{3}} \div -\frac{\sqrt{2}}{\sqrt{3}} = -\frac{1}{\sqrt{2}}$

4



First find AC. Angle ABC = 35 + 8 = 43°;

$AC^2 = 2.5^2 + 4.6^2 - 2 \times 2.5 \times 4.6 \times \cos 43° = 10.589$

and AC = 3.254

Now find angle ACB : $\frac{\sin ACB}{2.5} = \frac{\sin 43°}{3.254}$

so sin ACB = 0.5240 and angle ACB = 31.6°

Angle ACD = 98 + 31.6 = 129.6° so

$AD^2 = 3.7^2 + 3.254^2 - 2 \times 3.7 \times 3.254 \times \cos 129.6 = 39.63$

and AD = 6.30 km (3 s.f.)

5 **a** Not saying that $\cos x°$ could be 0 on the fifth line (the left-hand side should have been factorised in the third line); the fraction on the last line is upside down.

b This line is correct: $-4\cos^2 x + 3\cos x = 0$; either

$\cos x = 0$ or $\cos x = \frac{3}{4}$

Therefore $x = 90°$ or $270°$ or $41.4°$ or $318.6°$

6 When $x = 40$, $\cos(40a + b)° = 0$ so $40a + b = 90$

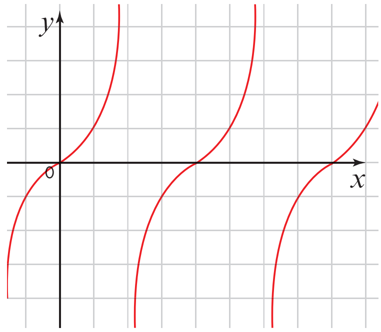
When $x = 100$, $\cos(100a + b)° = 0$ so this time

$100a + b = 270$

Subtract the equations to get $60a = 180$ and $a = 3$

Then $120 + b = 90$ so $b = -30$

7 The graph of $y = \tan kx$ looks like this:



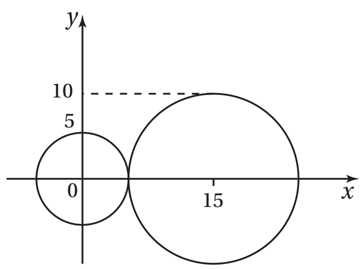
The period is $\frac{180}{k}$. If PQ = 10 then 10 must be a multiple of $\frac{180}{k}$.

That means $k = 10$ or $\frac{18}{2} = 9$ or $\frac{18}{3} = 6$ or generally $\frac{18}{n}$ where $n = 1, 2, 3, \ldots$

8 If $5 \sin 2x = 8 \cos 2x$ then $\tan 2x = 1.6$

Therefore $2x = 58$ or 238 or 418 or 598 or .... so $x = 29$ or 119 or 209 or 299

9 Put axes on the centre of disc A.



When disc A has rotated 500°, the coordinates of P are (5 cos 500°, 5 sin 500°) or (–3.830, 3.214).

Then disc B has rotated 250° and the coordinates of Q are (15 – 10 cos 250°, 10 sin 250°) or (18.420, –9.397).

Use Pythagoras to find the distance between these points:

$PQ^2 = (-3.830 - 18.420)^2 + (3.214 + 9.397)^2 = 654.1$

so PQ = 25.6

10 $BD3^2 = 21^2 + 24^2 - 2 \times 21 \times 24 \cos 70$

BD = 25.93

$\cos C = \frac{17^2 + 28^2 - 25.93^2}{2 \times 17 \times 28} = 0.4208$ so $C = 65.1°$

Now find angle ABD: $\frac{\sin ABD}{24} = \frac{\sin 70°}{25.93}$

so sin ABD = 0.8698 and angle ABD = 60.4°

Now find angle CBD: $\frac{\sin CBD}{28} = \frac{\sin 65.1°}{25.93}$

so sin CBD = 0.9795 and angle CBD = 78.4°

Angle $B = 60.4 + 78.4 = 138.8°$ and

angle $D = 360 - (70 + 138.8 + 65.1) = 86.1°$

11 Find angle $B: \frac{1}{2}\times 15\times 20\ \sin B = 120$

so $\sin B = \frac{120}{150} = 0.8$ and $B = 53.1°$ or $180 - 53.1 = 126.9°$

So there are two possible answers.

If $B = 53.1°$, then

$AC^2 = 15^2 + 20^2 - 2\times 15\times 20\cos 53.1° = 264.75$

and AC = 16.3 cm

If B = 126.9°, then

$AC^2 = 15^2 + 20^2 - 2\times 15\times 20\cos 126.9° = 985.25$

and AC = 31.4 cm

12 Write $\tan x° = t$ and then $t^2 - t - 12 = 0$

Factorise: $(t-4)(t+3) = 0$ and $t = 4$ or -3

If $\tan x° = 4$ then $x = 76.0°$ or $256.0°$

if $\tan x° = -3$ then $x = 108.4°$ or $288.4°$

13 From triangle PQS,

$QS^2 = (2x)^2 + 8^2 - 2\times 2x\times 8\cos 60° = 4x^2 - 16x + 64$

From triangle RQS,

$QS^2 = x^2 + 12^2 - 2\times x\times 12\cos 60° = x^2 - 12x + 144$

Equate these: $4x^2 - 16x + 64 = x^2 - 12x + 144$

Rearrange: $3x^2 - 4x - 80 = 0$

Use the quadratic formula: $x = \frac{4\pm\sqrt{16+960}}{6}$

Take the positive root: $x = 5.87$

14 $\sin x° = 2(1-\sin^2 x°)$ or $2\sin^2 x° + \sin x° - 2 = 0$

Use the quadratic formula:

$\sin x = \frac{-1\pm\sqrt{1+16}}{4} = 0.7808$ or -1.28

Only the first value is relevant, so $x = 51.3°$

15 Write $\sin x° = s$ and then $5s + 9 = 12(1-s^2)$

Rearrange: $12s^2 + 5s - 3 = 0$

Factorise: $(3s-1)(4s+3) = 0$ and $s = \frac{1}{3}$ or $-\frac{3}{4}$

If $\sin x° = \frac{1}{3}$, then $x = 19.5$ or $160.5°$

If $\sin x° = -\frac{3}{4}$, then $x = 228.6$ or 311.4

Chapter 7 Exponentials and logarithms

1 **a** Let $\log_{n^2} a = x$. Then $a = (n^2)^x = n^{2x}$ so $\log_n a = 2x$ and $x = \frac{1}{2}\log_n a$.

b Let $\log_a b = y$ and $\log_b c = z$. Then $b = a^y$ and $c = b^z = (a^y)^z = a^{yz}$.

Hence $\log_a c = yz$ as required.

2 $\log y = \log k + (\log a)x$ so a graph of $\log y$ against x has gradient $\log a$.

x	1.5	2.7
y	172	131
$\log y$	2.236	2.117

The gradient is $\frac{2.117-2.236}{2.7-1.5} = -0.099 = \log a$ so

$a = 10^{-0.099}$, and $a = 0.80$

Then $172 = k\times 0.8^{1.5}$ and $k = 240$ (to 3 s.f.)

3 $y = 1.4^{0.5x+2} = 1.4^{0.5x}\times 1.4^2$

$= 1.96\times 1.4^{0.5x} = 1.96\times e^{(\ln 1.4)0.5x}$

$y = 1.96e^{0.168x}$

4 **a** $\log_5 40 = \log_5(5\times 8) = \log_5 5 + \log_5 2^3 = 1 + 3c$

b $\log_5 0.08 = \log_5(8\div 100)$

$= \log_5 8 - (\log_5 2^2 + \log_5 5^2)$

$= \log_5 2^3 - \log_5 2^2 - \log_5 5^2$

$= 3\log_5 2 - 2\log_5 2 - 2\log_5 5$

$= 3c - 2c - 2$

$= c - 2$

5 $y = 4e^{1.2x}$ is an exponential curve and the gradient is $1.2y$; if $1.2y = 6$ then $y = 5$

$5 = 4e^{1.2x}$; $e^{1.2x} = 1.25$; $1.2x = \ln 1.25$; $x = 0.186$ (to 3 d.p.)

6 $\log_8 x + \log_8 2x + \log_8 x^2 = \log_8(x\times 2x\times x^2)$

$= \log_8 2x^4 = \log_8 2 + \log_8 x^4$

$= \frac{1}{3} + 4\log_8 x$

Therefore $\frac{1}{3} + 4\log_8 x = 10$ and $\log_8 x = \frac{10-\frac{1}{3}}{4} = 2.4167$

So $x = 8^{2.4167} = 152.2$

7 **a** The second line, $y = 4e^{2x} + 4e^2$, should be

$y = 4e^{2x}\times 4e^2$

The fourth line is wrong because $4e^{2x} \neq e^{8x}$

b $y = 4e^{2x+2}$; $\ln y = \ln 4 + 2x + 2$; $\ln y = \ln 2^2 + 2x + 2$;

$\ln y = 2\ln 2 + 2x + 2$

Divide by 2 and rearrange: $0.5\ln y - \ln 2 - 1 = x$

$x = \ln y^{0.5} - \ln 2 - 1$

$x = \ln\frac{y^{0.5}}{2} - 1$, as required.

8 **a** $d = kt^n$ where k is a constant, so

$\log d = \log k + n\log t$. Using values from the table,

$\log 227.9 = \log k + n\log 1.88$ and

$\log 778.3 = \log k + n\log 11.86$

Subtract:

$\log 778.3 - \log 227.9 = n(\log 11.86 - \log 1.88)$ and

$n = 0.667$

Substitute to find k: $227.9 = k\times 1.88^{0.667}$ so $k = 149.6$.

Therefore $d = 1496t^{0.667}$

b For the Earth, $t = 1$ and so $d = 149.6$ km

9 $5^{2x}+6=5^{x+1}$ can be rearranged as $\left(5^x\right)^2+6=5\times5^x$ or $\left(5^x\right)^2-5\times5^x+6=0$

This is a quadratic in 5^x which can be factorised as $\left(5^x-3\right)\left(5^x-2\right)=0$

Either $5^x=3$ or $5^x=2$ so $x=\frac{\log3}{\log5}=0.683$ or $x=\frac{\log2}{\log5}=0.431$

10 When the mass is 0.01 g, $0.05\times0.97^t=0.01$; $0.97^t=0.2$; $t=\frac{\log0.2}{\log0.97}=52.84$

The mass is $0.05\times0.97^t=0.05\times e^{(\ln0.97)t}$ and the rate of change is ln 0.97 × the mass.

When $t=52.84$ this is $\ln0.97\times0.05\times0.97^{52.84}=-3.046\times10^{-4}$ so the mass is decreasing at the rate 3.046×10^{-3} g/hour.

11 a After 1 year, the value is $\left(1+\frac{p}{100}\right)A$

so $\left(1+\frac{p}{100}\right)A=Ae^k$ and $k=\ln\left(1+\frac{p}{100}\right)$.

b After 1 year, the value is $\left(1+\frac{p}{100}\right)^{12}A$ so $\left(1+\frac{p}{100}\right)^{12}A=Ae^k$ and

$e^k=\left(1+\frac{p}{100}\right)^{12}$ so $k=12\ln\left(1+\frac{p}{100}\right)$.

12 If $y=ka^x$ then $\log y=\log k+x\log a$ and the gradient of the graph is $\log a$.

$\log a=\frac{2.14-3.51}{35-5}=-0.04567$ so $a=10^{-0.0457}=0.9$

Substitute to find k: $3.51=\log k+5\times0.04567$

so $\log k=3.738$ and $k=10^{3.738}=5500$ (to 2 s.f.)

Therefore $y=5500(0.9)^x$

13 When $t=88$ the mass is $\frac{1}{2}m$ so $e^{-88k}=0.5$

so $-88k=\ln0.5$ and $k=-\frac{\ln0.5}{88}=7.877\times10^{-3}$

When the mass is $0.1m$ then $me^{-kt}=0.1m$

so $-kt=\ln0.1$ and $t=-\frac{\ln0.1}{7.877\times10^{-3}}=292$

The answer is 292 years.

14 a Assume the population in t years' time is $8.24\times k^t$ million.

For 10 years ago, $t=-10$ and $8.24\times k^{-10}=6.85$;

$k^{10}=\frac{8.24}{6.85}=1.2029$

and $k=\sqrt[10]{1.2029}=1.0186$. The population is predicted to be given by $p=8.24\times1.0186^t$.

b The rate of growth could change because of a change in the birth or death rates or because of emigration or immigration.

c If the increase is 12% over 10 years then $k^{10}=1.12$ and $k=\sqrt[10]{1.12}=1.0114$ and the new formula is $p=8.24\times1.0114^t$. The value 8.24 does not change.

15 $e^x+e^{-x}=4$. Multiply by e^x: $e^{2x}+1=4e^x$

Rearrange: $e^{2x}-4e^x+1=0$

This is a quadratic in e^x so use the quadratic formula:

$e^x=\frac{4\pm\sqrt{16-4}}{2}=\frac{4\pm\sqrt{12}}{2}=2\pm\sqrt3$

Either $e^x=2+\sqrt3$ or $e^x=2-\sqrt3$; $x=\ln(2+\sqrt3)$ or $\ln(2-\sqrt3)$; $x=1.317$ or -1.3170

Chapter 8 Differentiation

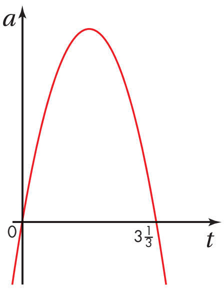
1 a $v=5x^2-x^3$ so $\frac{dv}{dx}=10x-3x^2$

If the acceleration is 0, then $\frac{dv}{dx}=0$.

$10x-3x^2=0\Rightarrow x(10-3x)+0\Rightarrow x=0$ or $3\frac{1}{3}$

The acceleration is 0 initially, and again after $3\frac{1}{3}$ s.

b



2 $y=4x\left(x^{\frac{3}{2}}-2x^{\frac{1}{2}}\right)=4x^{\frac{5}{2}}-8x^{\frac{3}{2}}$ so $\frac{dy}{dx}=10x^{\frac{3}{2}}-12x^{\frac{1}{2}}$.

3 $\frac{dy}{dx}=6x^2-6x-36$

at a stationary point, $\frac{dy}{dx}=0$ so $6x^2-6x-36=0$.

$x^2-x-6=0$; $(x-3)(x+2)=0$ so $x=3$ or -2

So there are two stationary points.

When $x=3$, $y=2\times27-3\times9-36\times3+10=-71$ so one stationary point is (3, –71).

$\frac{d^2y}{dx^2}=12x-6$, , which is 30 and positive at this point, so it is a minimum point.

When $x=-2$, $y=(2\times-8)-3\times4-(3\times6-2)+10=54$ so the other stationary point is (–2, 54).

$\frac{d^2y}{dx^2}=12x-6$,which is –30 and negative at this point, so it is a maximum point.

4 $y=x\left(2x^2+5x-12\right)=2x^3+5x^2-12x$

$\frac{dy}{dx}=6x^2+10x-12$

The graph crosses the x-axis where $y=0$, so $x=0$, 1.5 or –4

At (0, 0), $\frac{dy}{dx}=-12$

At (1.5, 0), $\frac{dy}{dx}=6\times1.5^2+15-12=16.5$

At (–4, 0), $\frac{dy}{dx}=6(-4)^2-40-12=44$

5 **a** The second line is incorrect: you cannot divide the numerator and denominator separately in that way.

b $y = \dfrac{x^2+3x}{x^{\frac{1}{3}}}$ so $y = x^{\frac{5}{3}} + 3x^{\frac{2}{3}}$

$\dfrac{dy}{dx} = \dfrac{5}{3}x^{\frac{2}{3}} + 2x^{-\frac{1}{3}} = \dfrac{\frac{5}{3}x+2}{x^{\frac{1}{3}}} = \dfrac{5x+6}{3\sqrt[3]{x}}$

6 **a** Volume $= x^2h = 500$ so $h = \dfrac{500}{x^2}$

The surface area is

$A = 2x^2 + 4xh = 2x^2 + 4x\dfrac{500}{x^2} = 2x^2 + \dfrac{2000}{x}$.

b $\dfrac{dA}{dx} = 4x - \dfrac{2000}{x^2}$

At a minimum point $\dfrac{dA}{dx} = 0$

so $4x - \dfrac{2000}{x^2} = 0$ and $x^3 = 500$

Since this is the volume, it implies that h is the same as x so the shape is a cube.

$\dfrac{d^2A}{dx^2} = 4 + \dfrac{4000}{x^3} = 12$, which is positive, showing that the surface area is a maximum.

7 $f(x) = \sqrt{2x} = \sqrt{2}x^{\frac{1}{2}}$

$f'(x) = \sqrt{2} \times \dfrac{1}{2}x^{-\frac{1}{2}} = \dfrac{1}{\sqrt{2}}x^{-\frac{1}{2}}$

$f''(x) = -\dfrac{1}{2} \times \dfrac{1}{\sqrt{2}}x^{-\frac{3}{2}}$

Now $\{f(x)\}^3 = \left\{\sqrt{2}x^{\frac{1}{2}}\right\}^3 = \left(\sqrt{2}\right)^3 x^{\frac{3}{2}}$

so $f''(x)\{f(x)\}^3 = -\dfrac{1}{2} \times \dfrac{1}{\sqrt{2}}x^{-\frac{3}{2}} \times \left(\sqrt{2}\right)^3 x^{\frac{3}{2}} = -1$

8 $\dfrac{dy}{dx} = 3x^2 - 6x - 6$

When $x = 3$, $y = 27 - 27 - 18 + 10 = -8$

and $\dfrac{dy}{dx} = 27 - 18 - 6 = 3$

The equation of the tangent is $y + 8 = 3(x - 3)$ or $y = 3x - 17$. On the tangent, when $x = 7$, $y = 4$, which shows that (7, 4) is on the tangent.

9 $y = 2x^2 + 3x + 1$

$y + \delta y = 2(x + \delta x)^2 + 3(x + \delta x) + 1 =$

$= 2x^2 + 4x\delta x + 2(\delta x)^2 + 3x + 3\delta x + 1$

$\delta y = (y + \delta y) - y = 4x\delta x + 2(\delta x)^2 + 3\delta x$

$\dfrac{\delta y}{\delta x} = \dfrac{4x\delta x + 2(\delta x)^2 + 3\delta x}{\delta x} = 4x + 2\delta x + 3$

Then $\dfrac{dy}{dx} = \lim_{\delta x \to 0} \dfrac{\delta y}{\delta x} = 4x + 3$.

10 $\dfrac{dy}{dx} = 2x$. If P is the point (x, y) then it must be the case that $\dfrac{y}{x} = 2x$ so $y = 2x^2$.

Substitute this into the equation:

$2x^2 = x^2 + 9 \Rightarrow x^2 = 9$ and so $x = 3$ as P is positive.

P is (3, 18) and the gradient of the tangent is 6, so the gradient of the normal is $-\dfrac{1}{6}$.

The equation of the normal is $y - 18 = -\dfrac{1}{6}(x - 3)$ and this crosses the x-axis where $y = 0$.

So $-18 = -\dfrac{1}{6}(x - 3)$. Therefore $x - 3 = 108$ and $x = 111$; the crossing point is (111, 0).

11 The volume is given by

$V = x(30 - 2x)^2 = x\left(900 - 120x + 4x^2\right.$

$= 900x - 120x^2 + 4x^3$

$\dfrac{dV}{dx} = 900 - 240x + 12x^2$

The maximum volume is when $\dfrac{dV}{dx} = 0$.

$900 - 240x + 12x^2 = 0$

$\rightarrow 75 - 20x + x^2 = 0$

$\rightarrow (x - 15)(x - 5) = 0$

and $x = 15$ or 5.

If $x = 15$ the volume is 0, so we want $x = 5$, and then the volume is $520^2 = 2000\text{ cm}^3$.

Check: $\dfrac{d^2y}{dx^2} = -240 + 24x$; when $x = 5$, $\dfrac{d^2y}{dx^2} = -120$, which is negative, so the volume is a maximum.

12 Volume $= V = \dfrac{1}{3}\pi r^2 h$

If $h + r = 60$ then $h = 60 - r$ and $V = \dfrac{1}{3}\pi r^2(60 - r)$.

$V = \dfrac{1}{3}\pi\left(60r^2 - r^3\right)$

Therefore $\dfrac{dV}{dr} = \dfrac{1}{3}\pi\left(120r - 3r^2\right)$.

When the volume is a maximum, $\dfrac{dV}{dr} = 0$

so $\dfrac{1}{3}\pi(120r - 3r^2) \Rightarrow 40r - r^2 + 0 \Rightarrow r(40 - r) = 0$ and so $r = 0$ or 40.

$r = 0$ gives a volume of 0, so $r = 40$ for a maximum volume and then $h = 60 - r = 20$.

The ratio $r : h = 40 : 20 = 2 : 1$

$\dfrac{d^2V}{dr^2} = \dfrac{1}{3}\pi(120 - 6r)$, which is negative when $r = 40$, confirming that the volume is a maximum.

13 $y = x^2 + a^2x^{-2}$ so $\dfrac{dy}{dx} = 2x - 2a^2x^{-3}$ or $\dfrac{dy}{dx} = 2x - \dfrac{2a^2}{x^3}$

At a stationary point, $\dfrac{dy}{dx} = 0$ so $2x - \dfrac{2a^2}{x^3} = 0$ and $x^4 = a^2$.

$x = \sqrt{a}$ or $-\sqrt{a}$.

When $x = \sqrt{a}$, then $y = a + a^2a^{-1} = a + a = 2a$

When $x = -\sqrt{a}$, then $y = a + a = 2a$.

The turning points are $(\sqrt{a}, 2a)$ and $(-\sqrt{a}, 2a)$.

14 $y=36x^{-2}$ so $\frac{\mathrm{d}y}{\mathrm{d}x}=36\times -2x^{-3}=-\frac{72}{x^3}$.

At (6, 1), $\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{72}{216}=-\frac{1}{3}$ so the gradient of the normal is 3.

The equation of the normal is $y-1=3(x-6)$.

Where it crosses the y-axis, $x=0$ and $y-1=-18$ so $y=-17$.

Where it crosses the x-axis, $y=0$ and $-1=3(x-6)$ so $x=5\frac{2}{3}$.

The area of the triangle is $\frac{1}{2}\times 17\times 5\frac{2}{3}=48\frac{1}{6}$.

15 a $\frac{1}{x+a}-\frac{1}{x}=\frac{x-(x+a)}{(x+a)x}=\frac{x-x-a}{x(x+a)}=-\frac{a}{x(x+a)}$

b If $\mathrm{f}(x)=\frac{1}{x}$, then $\mathrm{f}(x+\delta x)=\frac{1}{x+\delta x}$ and

$$\mathrm{f}(x+\delta x)-\mathrm{f}(x)=\frac{1}{x+\delta x}-\frac{1}{x}=-\frac{\delta x}{x(x+\delta x)}$$

Then $\frac{\delta y}{\delta x}=\frac{\mathrm{f}(x+\delta x)-\mathrm{f}(x)}{\delta x}=-\frac{1}{x(x+\delta x)}$

and $\frac{\mathrm{d}y}{\mathrm{d}x}=\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=-\frac{1}{x\times x}=-\frac{1}{x^2}$

Chapter 9 Integration

1 $\int\frac{(x-3)^2}{\sqrt{x}}\,\mathrm{d}x=\int\frac{x^2-6x+9}{x^{\frac{1}{2}}}\,\mathrm{d}x=\int x^{\frac{3}{2}}-6x^{\frac{1}{2}}+9x^{-\frac{1}{2}}\mathrm{d}x$

$=\frac{2}{5}x^{\frac{5}{2}}-4x^{\frac{3}{2}}+18x^{\frac{1}{2}}+c$

2 $\mathrm{f}(x)=\int\frac{1}{4}-9x^{-2}\mathrm{d}x=\frac{1}{4}x+9x^{-1}=\frac{1}{4}x+\frac{9}{x}+c$

$\mathrm{f}(4)=3.25$ so $\frac{1}{4}\times 4+\frac{9}{4}+c=3.25$; $3.25+c=5$; $c=1.75$

$\mathrm{f}(x)=\frac{1}{4}x+\frac{9}{x}+1.75$ and $k=\mathrm{f}(9)=2.25+1+1.75=5$

3 $\int\left(2x+\frac{3}{x}\right)^2\mathrm{d}x=\int 4x^2+12+9x^{-2}\mathrm{d}x$

$=\frac{4}{3}x^3+12x-9x^{-1}+c$

or $\frac{4}{3}x^3+12x-\frac{9}{x}+c$

4 Where the curve crosses the x-axis, $x^2+x-6=0$.

Factorise: $(x+3)(x-2)=0$

Therefore $x=-3$ or 2

$\int_{-3}^{2}x^2+x-6dx=\left[\frac{1}{3}x^3+\frac{1}{2}x^2-6x\right]_{-3}^{2}$

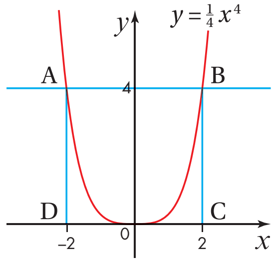
$=\left[-7\frac{1}{3}\right]-\left[13\frac{1}{2}\right]=-20\frac{5}{6}$

The area is positive and is $20\frac{5}{6}$

5 $\frac{\mathrm{d}y}{\mathrm{d}x}=2x-\frac{2}{x^3}$

Integrate: $y=\int 2x-2x^{-3}dx=x^2+x^{-2}+c$ or $x^2+\frac{1}{x^2}+c$

6 Where the line crosses the curve, $\frac{1}{4}x^4=4$ so $x^4=16$ and $x=2$ or -2



Area under curve between A and B is

$\int_{-2}^{2}\frac{1}{4}x^4\mathrm{d}x=\left[\frac{1}{20}x^5\right]_{-2}^{2}=[1.6]-[-1.6]=3.2$

Area of ABCD = 4 × 4 = 16

so required area $=16-3.2=12.8$

7 $\int_a^{4a}x^{-\frac{1}{2}}\mathrm{d}x=\left[2x^{\frac{1}{2}}\right]_a^{4a}$

$=\left[2\sqrt{4a}\right]-\left[2\sqrt{a}\right]=2\times 2\sqrt{a}-2\sqrt{a}=2\sqrt{a}$

Therefore $2\sqrt{a}=8$ so $\sqrt{a}=4$ and $a=16$

8 Where the curve crosses the x-axis, $x^2-3x=0$ so $x(x-3)=0$ and $x=0$ or 3.

Find the area in two parts. For the area below the x-axis, you find

$\int_2^3 x^2-3x\,\mathrm{d}x=\left[\frac{1}{3}x^3-\frac{3}{2}x^2\right]_2^3=\left[9-\frac{27}{2}\right]-\left[\frac{8}{3}-6\right]$

$=\left[-4\frac{1}{2}\right]-\left[-3\frac{1}{3}\right]=-1\frac{1}{6}$

So the area below the x-axis is $1\frac{1}{6}$

The area above the x-axis is $\int_3^4 x^2-3xdx$

$=\left[\frac{1}{3}x^3-\frac{3}{2}x^2\right]_3^4=\left[\frac{64}{3}-24\right]-\left[9-\frac{27}{2}\right]$

$=\left[-2\frac{2}{3}\right]-\left[-4\frac{1}{2}\right]=1\frac{5}{6}$

So the total area is $1\frac{1}{6}+1\frac{5}{6}=3$

9 Where the line crosses the curve, $x=\sqrt{4x}$ so $x=2\sqrt{x}$ and $x=0$ or $\sqrt{x}=2$, so $x=4$

The area between the curve and the x-axis is

$\int_0^4 2x^{\frac{1}{2}}\mathrm{d}x=\left[\frac{4}{3}x^{\frac{3}{2}}\right]_0^4=\frac{4}{3}\times 8=\frac{32}{3}=10\frac{2}{3}$

The area of the triangle under the straight line is $\frac{1}{2}\times 4\times 4=8$

So the area between the line and the curve is $10\frac{2}{3}-8=2\frac{2}{3}$

10 a Area $= \int_5^a 100x^{-2}\,dx = \left[-100x^{-1}\right]_5^a$

$= \left[-\frac{100}{a}\right] - \left[-\frac{100}{5}\right] = 20 - \frac{100}{a}$

This is always less than 20 but if a is large then $\frac{100}{a}$ is small. The larger a is, the closer the area is to 20.

b If the equation is $y = \frac{100}{x^3}$ then the area is

$\int_5^a 100x^{-3}\,dx = \left[-50x^{-2}\right]_5^a = \left[-\frac{50}{a^2}\right] - \left[-\frac{50}{25}\right]$

$= 2 - \frac{100}{a}$

This is always less than 2.

The larger a is, the closer the area is to 2.

11 The curves cross where $\frac{1}{2}x^2 = \frac{1}{8}x^3$.

Either $x = 0$ or $\frac{1}{2} = \frac{1}{8}x$ so $x = 4$

The area under $y = \frac{1}{2}x^2$ is

$\int_0^4 \frac{1}{2}x^2\,dx = \left[\frac{1}{6}x^3\right]_0^4 = \frac{32}{3} = 10\frac{2}{3}$

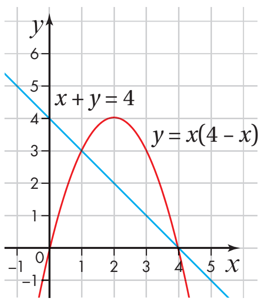
The area under $y = \frac{1}{8}x^3$ is $\int_0^4 \frac{1}{8}x^3\,dx = \left[\frac{1}{32}x^4\right]_0^4 = 8$

The area between the curves is $10\frac{2}{3} - 8 = 2\frac{2}{3}$

12 $\int \frac{x^2+1}{x^{\frac{1}{2}}} - \frac{x^2-1}{x^{\frac{3}{2}}}\,dx = \int x^{\frac{3}{2}} + x^{-\frac{1}{2}} - x^{\frac{1}{2}} + x^{-\frac{3}{2}}\,dx$

$= \frac{2}{5}x^{\frac{5}{2}} + 2x^{\frac{1}{2}} - \frac{2}{3}x^{\frac{3}{2}} - 2x^{-\frac{1}{2}} + c$

13



The equation of the straight line is $y = 4 - x$ and it crosses the curve where $4 - x = x(4 - x)$; $4 - x = 4x - x^2$; $x^2 - 5x + 4 = 0$; $(x-1)(x-4) = 0$; $x = 1$ or 4

The area under the curve is $\int_1^4 4x - x^2\,dx$

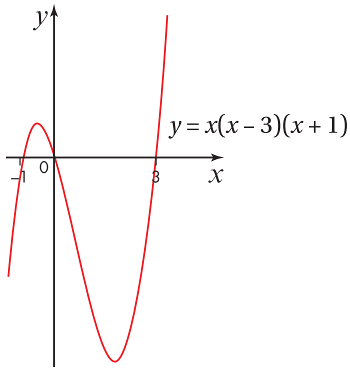
$= \left[2x^2 - \frac{1}{3}x^3\right]_1^4 = \left[32 - \frac{64}{3}\right] - \left[2 - \frac{1}{3}\right]$

$= 10\frac{2}{3} - 1\frac{2}{3} = 9$

The area of the triangle under the straight line is $\frac{1}{2} \times 3 \times 3 = 4.5$

The area between the curve and the line is 9 – 4.5 = 4.5

14 The curve crosses the x-axis at 0, 3 and –1.



Area between –1 and 0 is

$\int_{-1}^0 x^3 - 2x^2 - 3x\,dx = \left[\frac{1}{4}x^4 - \frac{2}{3}x^3 - \frac{3}{2}x^2\right]_{-1}^0$

$= [0] - \left[\frac{1}{4} + \frac{2}{3} - \frac{3}{2}\right] = \frac{7}{12}$

Area between 0 and 3 is

$\int_0^3 x^3 - 2x^2 - 3x\,dx = \left[\frac{1}{4}x^4 - \frac{2}{3}x^3 - \frac{3}{2}x^2\right]_0^3$

$= \left[\frac{81}{4} - 18 - \frac{27}{2}\right] - [0] = -11\frac{1}{4}$

The area below the x-axis is $11\frac{1}{4}$ so the total area is $\frac{7}{12} + 11\frac{1}{4} = 11\frac{5}{6}$

15 The equation can be written as $y = k(x+2)(x-4)$ so the integral you need is

$\int_{-2}^4 k\left(x^2 - 2x - 8\right)dx = \left[k\left(\frac{1}{3}x^3 - x^2 - 8x\right)\right]_{-2}^4$

$= k\left[\frac{64}{3} - 16 - 32\right] - k\left[\frac{-8}{3} - 4 + 16\right]$

$= k\left[-26\frac{2}{3}\right] - k\left[9\frac{1}{3}\right] = -36k$

Hence the area is $36k$ and so $36k = 108$ and $k = 3$.

The equation of the curve is $y = 3(x+2)(x-4)$.

Chapter 10 Vectors

1 a $|\mathbf{a}| = \sqrt{(-4\sqrt{3})^2 + 11^2} = \sqrt{48 + 121} = \sqrt{169} = 13$

b $\tan^{-1}\left[\frac{11}{4\sqrt{3}}\right] = 57.8°$

Required angle is 180° – 57.8° = 122.2°.

2 a 8 km south-west is represented by $\begin{bmatrix} -4\sqrt{2} \\ -4\sqrt{2} \end{bmatrix}$

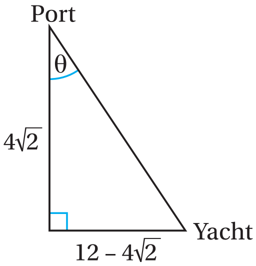
$\begin{bmatrix} 3 \\ 10 \end{bmatrix} + \begin{bmatrix} -4\sqrt{2} \\ -4\sqrt{2} \end{bmatrix} + \begin{bmatrix} 12 \\ 0 \end{bmatrix} = \begin{bmatrix} 15 - 4\sqrt{2} \\ 10 - 4\sqrt{2} \end{bmatrix}$

In the required form, this vector is

$\left(15 - 4\sqrt{2}\right)\mathbf{i} + \left(10 - 4\sqrt{2}\right)\mathbf{j}$

b The vector to return to port is

$$\begin{bmatrix} 3 \\ 10 \end{bmatrix} - \begin{bmatrix} 15-4\sqrt{2} \\ 10-4\sqrt{2} \end{bmatrix} = \begin{bmatrix} 4\sqrt{2}-12 \\ 4\sqrt{2} \end{bmatrix}$$



$$\theta = \tan^{-1}\left[\frac{12-4\sqrt{2}}{4\sqrt{2}}\right] = 48°$$

So the required bearing is 360° – 48° = 312°.

3 a $\overrightarrow{JK} = \overrightarrow{ML}$

$$\begin{bmatrix} 12 \\ -2 \end{bmatrix} - \begin{bmatrix} 2 \\ -9 \end{bmatrix} = \begin{bmatrix} a \\ 3 \end{bmatrix} - \begin{bmatrix} 5 \\ b \end{bmatrix}$$

$$\begin{bmatrix} 10 \\ 7 \end{bmatrix} = \begin{bmatrix} a-5 \\ 3-b \end{bmatrix}$$

$10 = a - 5$ so $a = 15$

$7 = 3 - b$ so $b = -4$

b $\overrightarrow{JK} = \begin{bmatrix} 10 \\ 7 \end{bmatrix}$

so angle between $\overrightarrow{JK}$ and **i** is

$\tan^{-1}(0.7) = 35°$.

$$\overrightarrow{JM} = \mathbf{m} - \mathbf{j} = \begin{bmatrix} 5 \\ -4 \end{bmatrix} - \begin{bmatrix} 2 \\ -9 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

so the angle between $\overrightarrow{JM}$ and **i** is $\tan^{-1}(1.\dot{6}) = 59°$.

Angle MJK = 59° – 35° = 24°

4 a $\mathbf{x} = \begin{bmatrix} -4 \\ 3 \end{bmatrix} + \begin{bmatrix} 15 \\ 9 \end{bmatrix} = \begin{bmatrix} 11 \\ 12 \end{bmatrix}$

$$\overrightarrow{XY} = \mathbf{y} - \mathbf{x} = \begin{bmatrix} 17 \\ 2 \end{bmatrix} - \begin{bmatrix} 11 \\ 12 \end{bmatrix} = \begin{bmatrix} 6 \\ -10 \end{bmatrix}$$

so $|\overrightarrow{XY}|^2 = 36 + 100 = 136$

$$\overrightarrow{WY} = \mathbf{y} - \mathbf{w} = \begin{bmatrix} 17 \\ 2 \end{bmatrix} - \begin{bmatrix} -4 \\ 3 \end{bmatrix} = \begin{bmatrix} 21 \\ -1 \end{bmatrix}$$

so $|\overrightarrow{WY}|^2 = 441 + 1 = 442$

$$\overrightarrow{WX} = \begin{bmatrix} 15 \\ 9 \end{bmatrix}$$

so $|\overrightarrow{WX}|^2 = 225 + 81 = 306$

306 + 136 = 442

Pythagoras' theorem holds, so the triangle is right angled.

b $\mathbf{z} = \mathbf{y} + \overrightarrow{XW} = \begin{bmatrix} 17 \\ 2 \end{bmatrix} + \begin{bmatrix} -15 \\ -9 \end{bmatrix} = \begin{bmatrix} 2 \\ -7 \end{bmatrix}$

c Area = WX × XY

$= \sqrt{306} \times \sqrt{136} = \sqrt{41\,616}$

$= 204$ (square units)

5 $\overrightarrow{PR} = \begin{bmatrix} -4 \\ a \end{bmatrix} - \begin{bmatrix} 5 \\ -20 \end{bmatrix} = \begin{bmatrix} -9 \\ 20+a \end{bmatrix}$

$\overrightarrow{RQ} = \begin{bmatrix} -1 \\ 16 \end{bmatrix} - \begin{bmatrix} -4 \\ a \end{bmatrix} = \begin{bmatrix} 3 \\ 16-a \end{bmatrix}$

$|\overrightarrow{PR}| = |\overrightarrow{RQ}|$ so

$$\sqrt{81+400+40a+a^2} = \sqrt{9+256-32a+a^2}$$

$$81 + 400 + 40a + a^2 = 9 + 256 - 32a + a^2$$

$$216 = -72a$$

$$a = -3$$

6 a $\overrightarrow{AB} = \begin{bmatrix} 3 \\ 16 \end{bmatrix} - \begin{bmatrix} 13 \\ 12 \end{bmatrix} = \begin{bmatrix} -10 \\ 4 \end{bmatrix}$

$\overrightarrow{BC} = \begin{bmatrix} -1 \\ 6 \end{bmatrix} - \begin{bmatrix} 3 \\ 16 \end{bmatrix} = \begin{bmatrix} -4 \\ -10 \end{bmatrix}$

$\overrightarrow{DC} = \begin{bmatrix} -1 \\ 6 \end{bmatrix} - \begin{bmatrix} 9 \\ 2 \end{bmatrix} = \begin{bmatrix} -10 \\ 4 \end{bmatrix}$

$\overrightarrow{AD} = \begin{bmatrix} 9 \\ 2 \end{bmatrix} - \begin{bmatrix} 13 \\ 12 \end{bmatrix} = \begin{bmatrix} -4 \\ -10 \end{bmatrix}$

$\overrightarrow{BC} = \overrightarrow{AD}$ and $\overrightarrow{AB} = \overrightarrow{DC}$, so the quadrilateral has two pairs of parallel and equal sides (all of magnitude $\sqrt{116}$). The quadrilateral is a rhombus.

To prove that it is a square, either check the diagonals are equal, or check one of the diagonals makes a right-angled triangle with two sides of the quadrilateral.

Diagonal AC $= \begin{bmatrix} -1 \\ 6 \end{bmatrix} - \begin{bmatrix} 13 \\ 12 \end{bmatrix} = \begin{bmatrix} -14 \\ -6 \end{bmatrix}$

$|\overrightarrow{AC}| = \sqrt{196+36} = \sqrt{232}$

$|\overrightarrow{AD}| = \sqrt{16+100}$

$|\overrightarrow{DC}| = \sqrt{16+100}$

As 116 + 116 = 232, the triangle ABC satisfies Pythagoras' theorem so there is a right angle at vertex B.

The quadrilateral is a square.

b $\mathbf{e} = \mathbf{a} + \frac{1}{2}\overrightarrow{AC} = \begin{bmatrix} 13 \\ 12 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} -14 \\ -6 \end{bmatrix} = \begin{bmatrix} 6 \\ 9 \end{bmatrix}$

$\overrightarrow{EB} = \mathbf{b} - \mathbf{e} = \begin{bmatrix} 3 \\ 16 \end{bmatrix} - \begin{bmatrix} 6 \\ 9 \end{bmatrix} = \begin{bmatrix} -3 \\ 7 \end{bmatrix}$

$\overrightarrow{AF} = \begin{bmatrix} -3 \\ 7 \end{bmatrix}$

$\mathbf{f} = \mathbf{a} + \overrightarrow{AF} = \begin{bmatrix} 13 \\ 12 \end{bmatrix} + \begin{bmatrix} -3 \\ 7 \end{bmatrix} = \begin{bmatrix} 10 \\ 19 \end{bmatrix}$

$\mathbf{f} = 10\mathbf{i} + 19\mathbf{j}$

7 **a** $\sqrt{9+a^2} = \sqrt{13}$, so $9+a^2 = 13$ and $a^2 = 4$

Given that $a < 0$, $a = -2$.

b $b\begin{bmatrix} 3 \\ -2 \end{bmatrix} + 4\begin{bmatrix} -4 \\ 7 \end{bmatrix} = \begin{bmatrix} 3b-16 \\ 28-2b \end{bmatrix}$

For this to be parallel to **i** the vertical component must be 0.

$28 - 2b = 0$

$b = 14$

8 **a** $\overrightarrow{XW} = \frac{1}{2}\left[\begin{bmatrix} 13 \\ -13 \end{bmatrix} + \begin{bmatrix} 7 \\ -8 \end{bmatrix}\right]$

$= \frac{1}{2}\begin{bmatrix} 20 \\ -21 \end{bmatrix} = \begin{bmatrix} 10 \\ -10.5 \end{bmatrix}$

$\mathbf{W} = \overrightarrow{OX} + \overrightarrow{XW} = \begin{bmatrix} -2 \\ -4 \end{bmatrix} + \begin{bmatrix} 10 \\ 10.5 \end{bmatrix} = \begin{bmatrix} 8 \\ -14.5 \end{bmatrix}$

b $|\overrightarrow{WX}| = \sqrt{10^2 + 10.5^2} = \sqrt{100 + 110.25}$

$= \sqrt{210.25} = 14.5 = \frac{29}{2}$

9 **a** $\overrightarrow{PQ} = \mathbf{q} - \mathbf{p} = \begin{bmatrix} 4 \\ 8 \end{bmatrix} - \begin{bmatrix} 3 \\ 14 \end{bmatrix} = \begin{bmatrix} 1 \\ -6 \end{bmatrix}$

$\overrightarrow{PS} = \mathbf{s} - \mathbf{p} = \begin{bmatrix} 9 \\ 15 \end{bmatrix} - \begin{bmatrix} 3 \\ 14 \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \end{bmatrix}$

$\overrightarrow{PQ}$ makes an angle of $\tan^{-1}(\frac{1}{6})$ with the negative direction of the y-axis.

$\overrightarrow{PS}$ makes an angle of $\tan^{-1}(\frac{1}{6})$ with the positive direction of the x-axis.

They are perpendicular to each other.

b $\overrightarrow{QR} = \mathbf{r} - \mathbf{q} = \begin{bmatrix} 17 \\ 4 \end{bmatrix} - \begin{bmatrix} 4 \\ 8 \end{bmatrix} = \begin{bmatrix} 13 \\ -4 \end{bmatrix}$

$\overrightarrow{SR} = \mathbf{r} - \mathbf{s} = \begin{bmatrix} 17 \\ 4 \end{bmatrix} - \begin{bmatrix} 9 \\ 15 \end{bmatrix} = \begin{bmatrix} 8 \\ -11 \end{bmatrix}$

$|\overrightarrow{QR}| = \sqrt{13^2 + 4^2} = \sqrt{169 + 16} = \sqrt{185}$

$|\overrightarrow{SR}| = \sqrt{8^2 + 11^2} = \sqrt{64 + 121} = \sqrt{185}$

c $|\overrightarrow{PQ}| = \sqrt{1^2 + (-6)^2} = \sqrt{37}$

$|\overrightarrow{PS}| = \sqrt{6^2 + 1^2} = \sqrt{37}$

$|\overrightarrow{PQ}| = |\overrightarrow{PS}|$ and $|\overrightarrow{QR}| = |\overrightarrow{SR}|$ therefore the quadrilateral has two pairs of equal adjacent sides, and so is a kite.

10 $\overrightarrow{BA} = \mathbf{a} - \mathbf{b} = \begin{bmatrix} -3 \\ 1 \end{bmatrix} - \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} -6 \\ -3 \end{bmatrix}$

D is at $\mathbf{c} + \overrightarrow{CD} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} + \begin{bmatrix} -6 \\ -3 \end{bmatrix} = \begin{bmatrix} -4 \\ -4 \end{bmatrix}$

11 $\overrightarrow{AB} = \mathbf{b} - \mathbf{a} = \begin{bmatrix} -2 \\ 3 \end{bmatrix} - \begin{bmatrix} -5 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$

$\overrightarrow{AC} = \mathbf{c} - \mathbf{a} = \begin{bmatrix} 1 \\ 6 \end{bmatrix} - \begin{bmatrix} -5 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \end{bmatrix} = 2\overrightarrow{AB}$

$\overrightarrow{AC}$ is a multiple of $\overrightarrow{AB}$ so they are parallel. Since they have a common point, they are collinear.

12 $\overrightarrow{AB} = \mathbf{b} - \mathbf{a} = \begin{bmatrix} 12 \\ 3 \end{bmatrix} - \begin{bmatrix} 2 \\ 13 \end{bmatrix} = \begin{bmatrix} 10 \\ -10 \end{bmatrix}$

$\mathbf{c} = \mathbf{a} + \frac{1}{5}\overrightarrow{AB} = \begin{bmatrix} 2 \\ 13 \end{bmatrix} + \frac{1}{5}\begin{bmatrix} 10 \\ -10 \end{bmatrix} =$

$= \begin{bmatrix} 2 \\ 13 \end{bmatrix} + \begin{bmatrix} 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 4 \\ 11 \end{bmatrix}$

13 $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$

$\overrightarrow{BC} = \mathbf{c} - \mathbf{b}$

$\overrightarrow{AC} = \mathbf{c} - \mathbf{a}$

A, B and C are collinear, therefore $\mathbf{b} - \mathbf{a} = s(\mathbf{c} - \mathbf{b}) = t(\mathbf{c} - \mathbf{a})$ for some constants s and t.

$s\mathbf{c} - s\mathbf{b} = t\mathbf{c} - t\mathbf{a}$

$t\mathbf{a} - s\mathbf{b} + s\mathbf{c} - t\mathbf{c} = 0$

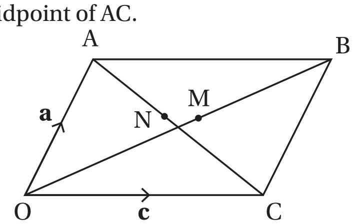
$t\mathbf{a} - s\mathbf{b} + (s - t)\mathbf{c} = 0$

$k\mathbf{a} + l\mathbf{b} + m\mathbf{c} = 0$

where $k + l + m = t + (-s) + (s - t) = t - s + s - t = 0$

14 In parallelogram OABC let **a** and **c** be the position vectors of A and C, respectively.

Let M be the midpoint of OB and let N be the midpoint of AC.



Position vector of N:

$\overrightarrow{AC} = \mathbf{c} - \mathbf{a}$

$\overrightarrow{AN} = \frac{1}{2}(\mathbf{c} - \mathbf{a})$ as N is the midpoint of AC

$\overrightarrow{ON} = \mathbf{a} + \frac{1}{2}(\mathbf{c} - \mathbf{a}) = \frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{c} = \frac{1}{2}(\mathbf{a} + \mathbf{c})$

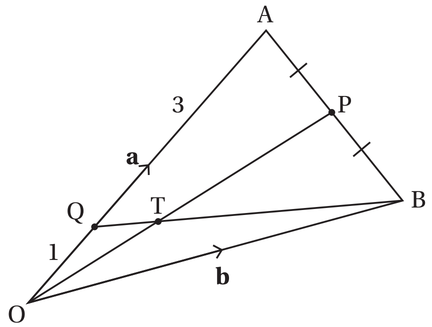
Position vector of M:

$\overrightarrow{OB} = \mathbf{a} + \mathbf{c}$

$\overrightarrow{OM} = \frac{1}{2}(\mathbf{a} + \mathbf{c})$ as M is the midpoint of OB

Since the midpoints have the same position vector, they must be the same point. Hence the diagonals bisect each other.

15 When a diagram is not given, it often helps to draw your own.



$\overrightarrow{OQ} = \frac{1}{4}\mathbf{a}$

$\overrightarrow{QB} = \mathbf{b} - \frac{1}{4}\mathbf{a}$

$\overrightarrow{OT} = \overrightarrow{OQ} + \overrightarrow{QT}$

$= \overrightarrow{OQ} + t\overrightarrow{QB}$

$= \overrightarrow{OQ} + t(\mathbf{b} - \frac{1}{4}\mathbf{a})$

$= \frac{1}{4}\mathbf{a} + t(\mathbf{b} - \frac{1}{4}\mathbf{a})$

$= \frac{1}{4}\mathbf{a} - \frac{1}{4}t\mathbf{a} + t\mathbf{b}$

$= (\frac{1}{4} - \frac{1}{4}t)\mathbf{a} + t\mathbf{b}$

OT lies on OP, so $\overrightarrow{OT}$ must be a scalar multiple of a + *b*

therefore $\frac{1}{4} - \frac{1}{4}t = t$

$\frac{1}{4} = t + \frac{1}{4}t$

$\frac{1}{4} = \frac{5}{4}t$

$t = \frac{1}{5}$

$\overrightarrow{OT} = \frac{1}{5}\mathbf{a} + \frac{1}{5}\mathbf{b}$

Chapter 11 Proof

1 **a** Use $2n + 1$ to represent any odd number, and square it.

$(2n+1)^2 = 4n^2 + 4n + 1 = 4(n^2 + n) + 1 = 4m + 1$

(replace $n^2 + n$ with m).

b Use a counter example, for example 2. Then $2^2 = 4$. This is a multiple of 4, not 1 more than a multiple of 4. (In fact, it can be proved that the square of any even number is a multiple of 4.)

2 Let the consecutive numbers be $(n - 1)$, n and $(n + 1)$.

$n^2 - (n-1)(n+1) = n^2 - (n^2 - 1) = n^2 - n^2 + 1 = 1$

3 Total surface area

= area of the triangular ends + area of the 3 rectangles

$= 2 \times \frac{1}{2}(2x)(x+3) + 4x(x+3) + 4x \times 2x + 4x(2x+3)$

$= 2x^2 + 6x + 4x^2 + 12x + 8x^2 + 8x^2 + 12x$

$= 22x^2 + 30x$

Given that the total surface area is 486 cm

$22x^2 + 30x = 486$

$11x^2 + 15x = 243$

$11x^2 + 15x - 243 = 0$, as required.

4 Complete the square:

$x^2 - 10x + 26 = (x-5)^2 + 1$

which is always positive.

5 Let δx be a small increase in x and let δy be the corresponding small increase in y.

$y = x^3$

$\Rightarrow \ y + \delta y = (x + \delta x)^3 = x^3 + 3x^2\delta x + 3x(\delta x)^2 + (\delta x)^3$

$\Rightarrow \quad \delta y = 3x^2\delta x + 3x(\delta x)^2 + (\delta x)^3$

$\Rightarrow \quad \frac{\delta y}{\delta x} = 3x^2 + 3x\delta x + (\delta x)^2$

and $\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = 3x^2$

6 Complete the square:

$y = 2x^2 - 8x + 28$

$= 2(x^2 - 4x + 14)$

$= 2((x-2)^2 + 10)$

$= 2(x-2)^2 + 20$

As $2(x-2)^2 \geqslant 0$, $2(x-2)^2 + 20 \geqslant 20$, i.e. $y \geqslant 20$

7 Use a sensible notation, such as 1 to indicate going up one step and 2 to indicate going up two steps. Work systematically.

Taking all 6 steps one at a time:

1, 1, 1, 1, 1, 1

Taking one pair of steps as a 'two' and all the rest one at a time:

2, 1, 1, 1, 1
1, 2, 1, 1, 1
1, 1, 2, 1, 1
1, 1, 1, 2, 1
1, 1, 1, 1, 2

Taking two pairs of steps as a 'two' and all the rest one at a time:

2, 2, 1, 1
2, 1, 2, 1
2, 1, 1, 2
1, 2, 2, 1
1, 2, 1, 2
1, 1, 2, 2

Taking all the steps in twos:

2, 2, 2

This lists all 13 possible ways of going up the stairs one or two steps at a time.

8 $2y - x - 4 = 0$

$$2y = x + 4$$

$$y = \frac{1}{2}x + 2$$

So, the gradient of the line is $\frac{1}{2}$.

Gradient of AB $= \frac{y_2 - y_1}{x_2 - x_1} = \frac{1-(-1)}{2-(-2)} = \frac{2}{4} = \frac{1}{2}$.

As the line, l, and AB have the same gradient, they are either parallel or the same line.

Check the coordinates of one of the points.

If (2, 1) lies on the line, then it will satisfy the equation of the line.

$2y - x - 4 = 2 \times 1 - 2 - 4 = 2 - 2 - 4 = -4$

As the coordinates of B do not satisfy the equation, AB is not the same line as l, and the two lines, AB and l are parallel.

9 Use $\sin^2\theta + \cos^2\theta = 1$.

$$\frac{\cos^2\theta}{1-\sin\theta} = \frac{1-\sin^2\theta}{1-\sin\theta} = \frac{(1-\sin\theta)(1+\sin\theta)}{1-\sin\theta} = 1 + \sin\theta$$

10 First find the centre and radius of the circle. To do this, complete the square twice.

$$x^2 - 2x + y^2 - 4y = 20$$

$$(x-1)^2 - 1 + (y-2)^2 - 4 = 20$$

$$(x-1)^2 + (y-2)^2 = 25$$

Circle centre is (1, 2) and radius is 5 units.

If (5, 6) is more than 5 units from the circle centre, then it lies outside the circle. Let d be the distance from the centre of the circle to (5, 6).

$d = \sqrt{(6-2)^2 + (5-1)^2} = \sqrt{16+16} = \sqrt{32}$

$\sqrt{32} > 5$ therefore (5, 6) lies outside the circle.

11 **a** Let $n = 2k + 1$.

Then $n^2 - 1 = (2k+1)^2 - 1 = 4k^2 + 4k + 1 - 1$

$$= 4k^2 + 4k = 4k(k+1)$$

As k and $k + 1$ are consecutive numbers, one of them must be even, i.e. a multiple of 2.

$4k(k+1)$ is a product of 4 and an even number so it must be a multiple of 8.

b Counter example, $n = 2$: $n^2 - 1 = 4 - 1 = 3$.

This is not a multiple of 8.

12 $m^3 - m = m(m^2 - 1) = m(m+1)(m-1)$

$$= (m-1)m(m+1)$$

This is the product of three consecutive numbers.

At least one of these numbers must be even and one must be a multiple of 3.

The product of an even number with a multiple of 3 is a multiple of 6, i.e. $(m-1)m(m+1)$ is a multiple of 6.

Therefore $m^3 - m$ must also be a multiple of 6.

13 Let $a = x,\ b = x+1,\ c = x+2,\ d = x+3$.

$$bd - ac = (x+1)(x+3) - x(x+2)$$
$$= x^2 + 4x + 3 - (x^2 + 2x)$$
$$= x^2 + 4x + 3 - x^2 - 2x$$
$$= 2x + 3$$
$$b + c = x + 1 + x + 2$$
$$= 2x + 3$$

$bd - ac = b + c$, as required.

14 Let $a = x,\ b = x+1,\ c = x+2,\ d = x+3$

$$bc - ad = (x+1)(x+2) - x(x+3)$$
$$= (x^2 + 3x + 2) - (x^2 + 3x)$$
$$= x^2 + 3x + 2 - x^2 - 3x$$
$$= 2$$

$bc - ad = 2$, as required.

15 To prove the triangle is right angled, you need to show that the sides satisfy Pythagoras' theorem.

$$(2x)^2 = 4x^2$$
$$(x^2+1)^2 = x^4 + 2x^2 + 1$$
$$(x^2-1)^2 = x^4 - 2x^2 + 1$$
$$(x^4 - 2x^2 + 1) + (4x^2) = x^4 + 2x^2 + 1$$
$$(x^2-1)^2 + (2x)^2 = (x^2+1)^2$$

Pythagoras' theorem is satisfied and the hypotenuse is the side with length $x^2 + 1$.

16 Let the numbers be x and $x + 3$.

The difference between the squares of these numbers is $(x + 3)^2 - x^2 = x^2 + 6x + 9 - x^2$

$$= 6x + 9$$
$$= 3(2x + 3)$$

$3(2x+3)$ is a multiple of 3, so the difference between the squares of the numbers is also a multiple of 3, as required.

Chapter 12 Data presentation and interpretation

1 $y = \frac{(x - 50\,000)}{2500}$

$24 \times 2500 = x - 50\,000$

$60\,000 = x - 50\,000$

$x = £110\,000$

2 **a** $\bar{x} = \frac{2176}{60} = 36.3$

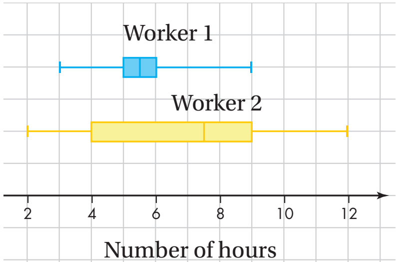
b $\sigma = \sqrt{\frac{81\,429.5}{60} - \left(\frac{2176}{60}\right)^2} = 6.47$

c 30.5th value, which is in 35–39 category.

Median $= 34.5 + \frac{12.5}{21} \times 5 = 37.5$ minutes

3 **a** Median 1 = 5.5

b LQ = 5, UQ = 6 so IQR = 1

c 

d Worker 1 has a lower median of 5.5 compared with 7.5 for worker 2. Worker 1 has an IQR of 1 compared with 5 for worker 2. Worker 1 is better to employ as they are faster and more consistent.

e

	Worker 1	**Worker 2**
LQ – 1.5 × IQR	5 – 1.5 × 1 = 3.5 so 3 is an outlier	4 – 1.5 × 5 = –3.5 no outliers
UQ + 1.5 × IQR	6 + 1.5 × 1 = 7.5 so 9 is an outlier	9 + 1.5 × 5 = 16.5 no outliers

4 a $\bar{x} = \frac{640}{50} = 12.8\text{ g}$

$\sigma = \sqrt{\frac{9341}{50} - \left(\frac{640}{50}\right)^2} = 4.79\text{ g}$

b Brand R has a more consistent mass and the mean is greater – this would be the better brand to buy.

c $20 \times 13.7 = 274\text{ g}$

$\bar{x} = \frac{274 + 640}{70} = 13.06\text{ g}$

$1.3 = \sqrt{\frac{\sum x^2}{20} - 13.7^2}$

$\sum x^2 = 3787.6$

Total = 3787.6 + 9341 = 13 128.6

$\sigma = \sqrt{\frac{13\,128.6}{70} - 13.06^2} = 4.12\text{ g}$

5 a Census – all members of the population are used.

b Sample – a selection of the population.

c Select 45 tyres using a random number generator based on the unique codes on the tyre wall, forming a random sample.

d A sample is quick, easy and cheap. A population is time consuming, expensive and unrealistic.

6 a $\bar{x} = \frac{789}{280} = 2.82\text{ kg}$

b $\sigma = \sqrt{\frac{2402.875}{280} - \left(\frac{789}{280}\right)^2} = 0.801\text{ kg}$

c 140.5th value, which is in the $2.5 < w \le 3$ category.

Median $= 2.5 + \frac{74.5}{99} \times 0.5 = 2.88\text{ kg}$

7 a $\frac{h_2}{h_1} = \frac{\text{frequency density}_2}{\text{frequency density}_1}$

$h_2 = \frac{5.4}{1.2} \times 3 = 13.5\text{ cm}$

$\frac{w_2}{w_1} = \frac{\text{class width}_2}{\text{class width}_1}$

$w_2 = \frac{5}{10} \times 2 = 1\text{ cm}$

b 58th value, which is in the 21–25 category.

Median $= 20.5 + \frac{27}{47} \times 5 = 23.4\text{s}$

c $\bar{x} = \frac{2585.5}{115} = 22.48\text{s}$

$\sigma = \sqrt{\frac{66\,448.75}{115} - \left(\frac{2585.5}{115}\right)^2} = 8.51\text{s}$

8 a 11–12 has a true class width of 2 cm, and 16–20 has a true class width of 5 cm.

Therefore width of block is 10 cm.

Frequency density of 11–12 is $\frac{24}{2} = 12$ pencils per cm.

Frequency density of 16–20 is $\frac{10}{5} = 2$.

Therefore height of 16–20 block is $\frac{24}{6} = 4\text{ cm}$.

b Cumulative frequency of median is 40.

Maximum length	8.5	10.5	12.5
Cumulative frequency	11	25	49

Therefore median $\approx 10.5 + 15 \times \frac{2}{24} \approx 11.75\text{ cm}$.

c 5.5 cm

d

Midpoint, *x*	5.5	9.5	11.5	13	14.5	18
f	11	14	24	10	10	10

$\sum xf = 924.5$, therefore mean $\approx 11.7\text{ cm}$.

$\sum x^2 f = 11\,802.75$

$\sigma \approx \sqrt{\frac{11802.75}{79} - \left(\frac{924.5}{79}\right)^2} = 3.53\text{ cm}$

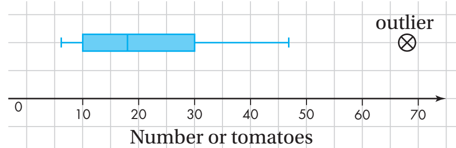
9 a Median is the 13th item, 18 tomatoes per plant.

b Quartiles are 6.5th items from either extreme, so $Q_3 = 30$, $Q_1 = 10$.

c Outliers are items, x, where

$x < 10 - 1.5\,(30 - 10) = -20$, so no outliers, or

$x > 30 + 1.5\,(30 - 10) = 60$, therefore 68 is an outlier.



10 a $\bar{x} = \frac{4967.8}{10} = 496.78\text{ g}$

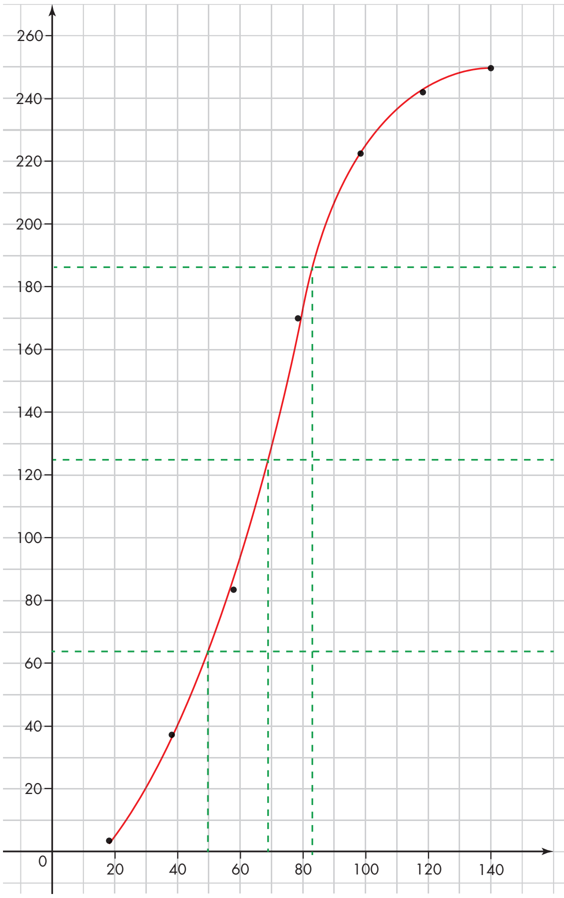
$\sigma^2 = 126.13\text{ g}$

b $\bar{x} = \frac{4897.3}{10} = 489.73\text{ g}$

$\sigma^2 = \frac{2399040.4}{10} - \left(\frac{4897.3}{10}\right)^2 = 68.57\text{ g}$

c Sample 1 has a higher mean than sample 2.
Sample 1 has a larger variance than sample 2.
Although sample 1 has a higher mean, there is less consistency than in sample 2.

11 a



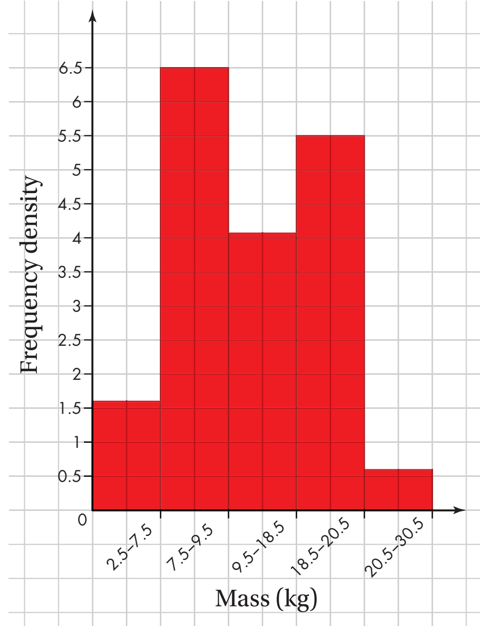
b Median about 70 marks

90th percentile about 107 marks

c 70% of 250 is 175, this is about 83 marks.

The pass mark should be 83 marks.

12



Mass (kg)	3–7	8–9	10–18	19–20	21–30
Frequency	8	13	37	11	6
Frequency density	1.6	6.5	4.1	5.5	0.6

b Median = 38th value

$9.5 + \frac{17}{37} \times 10 = 14.1\,\text{kg}$

13 $\frac{(v + 250)}{0.05} = 1.8(t + 75) + 72.1$

$v + 250 = 0.05\,(1.8t + 135) + 0.05\,(72.1)$

$v + 250 = 0.09t + 6.75 + 3.605$

$v + 250 = 0.09t + 10.355$

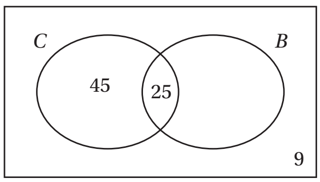
$v = 0.09t - 239.645$

Chapter 13 Probability and statistical distributions

1 a $45 + 25 + 9 = 69$

$100 - 69 = 21$

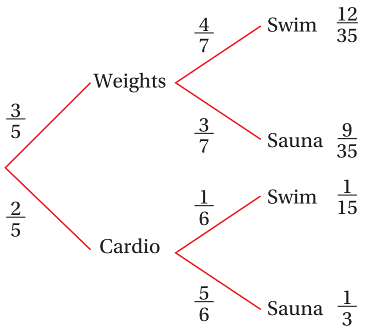
$21 + 45 = 66\%$ eat burger only and 45% eat chips only so 66% eat burger or chips but not both.



b $25 + 21 = 46\%$

$\frac{25}{46} = 54.3\%$

2 **a**



$\frac{12}{35} + \frac{1}{3} = \frac{71}{105}$

b $\frac{2}{5}$

c $\text{P(sauna)} = \frac{9}{35} + \frac{1}{3} = \frac{62}{105}$

$\text{P(weights | sauna)} = \dfrac{\frac{9}{35}}{\frac{62}{105}} = \frac{27}{62}$

3 **a**

X	2	4	6	8	10	12	14	16	18
$\mathbf{P}(X = x)$	$\frac{1}{64}$	$\frac{4}{64}$	$\frac{10}{64}$	$\frac{14}{64}$	$\frac{15}{64}$	$\frac{10}{64}$	$\frac{7}{64}$	$\frac{2}{64}$	$\frac{1}{64}$

b $P(X < 10) = \frac{29}{64}$

4 **a**

X	1	2	3	4	5	6
$\mathbf{P}(X = x)$	$\frac{1}{21}$	$\frac{2}{21}$	$\frac{3}{21}$	$\frac{4}{21}$	$\frac{5}{21}$	$\frac{6}{21}$
$\mathbf{F}(x)$	$\frac{1}{21}$	$\frac{3}{21}$	$\frac{6}{21}$	$\frac{10}{21}$	$\frac{15}{21}$	$\frac{21}{21}$

b $P(X \leqslant 4) = \frac{10}{21}$

c $P(2 < X \leqslant 3) = \frac{3}{21}$

5 **a** $P(X < 3) = 0.05126$

b $P(X > 5) = 1 - P(X \leqslant 5)$

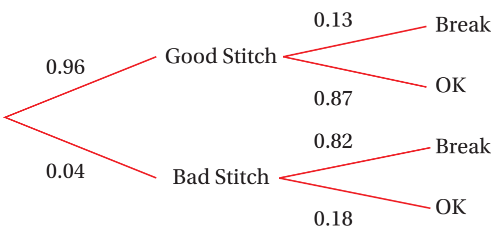
$= 1 - 0.4353 = 0.5647$

c $P(4) = 0.1334$

d $50 \times 12 = 600$ packets

$600 \times 0.12 = 72$

6 **a**



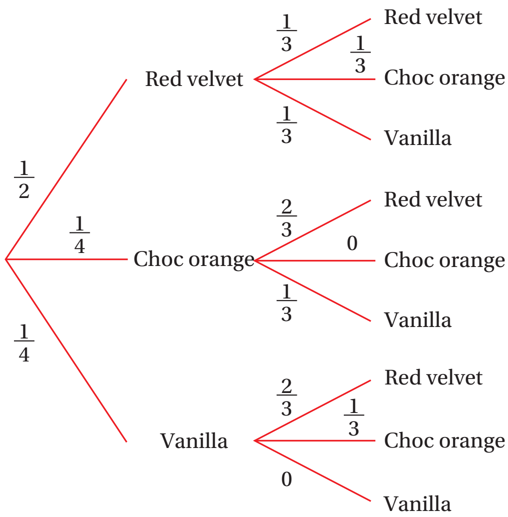
b $0.96 \times 0.13 + 0.04 \times 0.18 = 0.132$

c $0.96 \times 0.87 \times 0.89 = 0.743$

d $0.96 \times 0.87 \times 0.11 + 0.96 \times 0.13 \times 0.89$

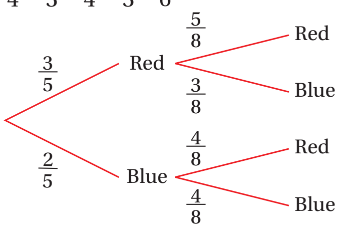
$+ 0.04 \times 0.18 \times 0.89 = 0.209$

7 **a**



b $\frac{1}{4} \times \frac{1}{3} + \frac{1}{4} \times \frac{1}{3} = \frac{1}{6}$

8 **a**



b $\text{P(red)} = \text{P(red}_A \text{ and red}_B) + \text{P(blue}_A \text{ and red}_B)$

$= \frac{3}{5} \times \frac{5}{8} + \frac{2}{5} \times \frac{4}{8} = \frac{23}{40}$

$\text{P}(\text{red}_A | \text{red}) = \dfrac{\text{P}(\text{red}_A \text{ and red}_B)}{\text{P(red)}}$

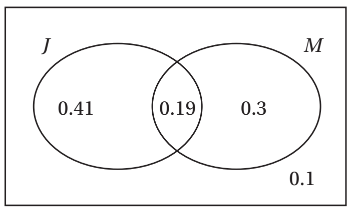
$= \dfrac{\frac{3}{5} \times \frac{5}{8}}{\frac{23}{40}} = \frac{15}{23}$

9 $\Sigma P(X = x) = 1 \Rightarrow 3a + 0.7 = 1 \Rightarrow a = 0.1$

10 **a** $P(J) = 0.6$

b $P(J \cap M) = 0.19$

c



d $P(J' \cap M') = 0.1$

e If J and M are independent, $P(J \cap M) = P(J) \times P(M)$

$0.19 \neq 0.6 \times 0.49$ so not independent.

11 **a** $P(X < 4) = \frac{3}{15}$

b $P(2 < X \leqslant 11) = \frac{9}{15} = \frac{3}{5}$

c $P(X > 13) = \frac{2}{15}$

12

X	0	1	2	3	4	5
$P(X = x)$	0.01	0.09	0.13	0.5	0.15	0.12

13 **a** $X \sim B(7, 0.8)$

b $P(X \geqslant 5) = 1 - P(X \leqslant 4) = 1 - 0.148 = 0.852$

14 **a** P(Exactly 1) = 0.0148

b P(<3 eggs) = 0.0733

c P(⩾ 5 eggs) = 1 – 0.41 = 0.59

15 **a** P(3) = 0.0781

b 6 × P(4) = 0.0938

16 $X \sim B(10, \frac{1}{3})$

a P(exactly 2) = 0.195

b $P(\leqslant 2) = P(0) + P(1) + P(2) = 0.299$

c $np = 10 \times \frac{1}{3} = 3\frac{1}{3}$

d P(3) = 0.26, P(4) = 0.228 therefore 3 most likely

Chapter 14 Statistical sampling and hypothesis testing

1 **a** $H_0: p = \frac{1}{5}$

$H_1: p > \frac{1}{5}$ where p is the probability of selecting the favourite coffee brand.

b $P(X \geqslant 5) = 1 - P(X < 5)$
$= 1 - 0.9672$
$= 0.0328$

As this is below 5%, evidence suggests that the student can tell the difference, so reject the null hypothesis.

2 $H_0: p = 0.57$
$H_1: p < 0.57$
where p is the probability of good value for money.
$P(X \leqslant 15) = 0.0102$
This is above 1% so evidence suggests that students still believe they are getting good value for money, so accept the null hypothesis.

3 $H_0: p = 0.16$
$H_1: p \neq 0.16$
where p is the probability of believing in Santa Claus.
Below 5%, $P(X \leqslant 9)$
Above 5%, $P(X \geqslant 22)$

4 $H_0: p = \frac{38}{60}$
$H_1: p > \frac{38}{60}$
where p is the probability of saying the same word in an hour.
$P(X \geqslant 43) = 1 - P(X < 42)$
$= 1 - 0.88728$
$= 0.11272$
As this is above 5%, evidence suggests that the teacher has not increased his usage of the word, so accept the null hypothesis.

5 $H_0: p = 0.55$
$H_1: p < 0.55$
where p is the probability of her being backed.
$P(X \leqslant 16) = 0.1019$
As this is above 10%, evidence suggests that the student has not over estimated, so accept the null hypothesis.

6 $H_0: p = 0.2$
$H_1: p \neq 0.2$
Below 5%, $P(X \leqslant 2) = 0.019$
Above 5%, $P(X \geqslant 12) = 0.0344$

7 $H_0: p = 0.63$
$H_1: p > 0.63$
where p is the probability of passing first time.
$P(X \geqslant 15) = 1 - P(X < 15)$
$= 1 - 0.9439$
$= 0.0561$
As this is above 5%, evidence suggests that the app hasn't improved, so accept the null hypothesis.

8 $H_0: p = 0.18$
$H_1: p < 0.18$
where p is the probability of mis-shapen lenses.
$P(X \leqslant 3) = 0.0542$
As this is above 5%, evidence suggests that the service has not improved the machine, so accept the null hypothesis.

9 **a** $H_0: p = 0.75$
$H_1: p < 0.75$
where p is the probability of increasing lung capacity.

b $P(X < 14) = 0.00521$
As this is below 5%, evidence suggests that the programme does improve lung capacity, so reject the null hypothesis. Because $\frac{14}{26}$ is about 53% you do not need to look higher than 75%.

c That the people in the sample are only doing this programme, and are not undertaking any other form of exercise or taking performance-enhancing medication.

10 a $H_0: p = 0.23$

$H_1: p < 0.23$

where p is the probability of giving up takeaways.

b Below 10%, $P(X \leqslant 7) = 0.08444$

c $P(X \leqslant 6) = 0.0397$

As this is below 5% in the lower tail, evidence suggests that young people are trying to give up, so reject the null hypothesis.

Alternatively: 6 is below 7, which is the critical value, therefore you know that this must be in the critical region.

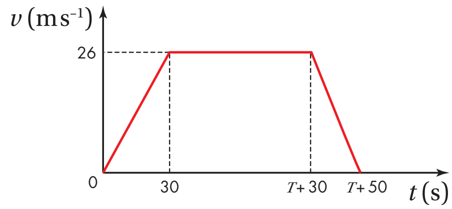
Chapter 15 Kinematics

1 $u = 53.9\,\text{m s}^{-1}$, $a = -9.8\,\text{m s}^{-2}$, $s = 88.2\,\text{m}$

$$s = ut + \tfrac{1}{2}at^2$$
$$88.2 = 53.9t - 4.9t^2$$
$$4.9t^2 - 53.9t + 88.2 = 0$$
$$t^2 - 11t + 18 = 0$$
$$(t-2)(t-9) = 0$$
$$t = 2 \text{ or } 9$$

Rocket is more than 88.2 m above the ground for 7 s.

2



$$\tfrac{1}{2}(T + T + 50) \times 26 = 910$$
$$2T + 50 = 70$$
$$T = 10$$

3 $r = (24t^2 - 12t - t^4 + 15)$ m.

Differentiating r with respect to t: $v = 48t - 12 - 4t^3$

Differentiating v with respect to t: $a = 48 - 12t^2$

For maximum velocity, $a = 0\,\text{m s}^{-2}$

$$48 - 12t^2 = 0$$
$$4 - t^2 = 0$$
$$(2 - t)(2 + t) = 0$$
$$t = 2\,\text{s}$$

When $t = 2$, $r = 24(2)^2 - 12(2) - (2)^4 + 15 = 71$ m.

4 a Athlete: $u = 0\,\text{m s}^{-1}$, $a = 0.8\,\text{m s}^{-2}$, $t = T$ s

$$s = ut + \tfrac{1}{2}at^2$$
$$s = 0.4T^2$$

Opponent: $u = 0\,\text{m s}^{-1}$, $a = 0.6\,\text{m s}^{-2}$, $t = (T+2)$ s

$$s = ut + \tfrac{1}{2}at^2$$
$$s = 0.3(T+2)^2$$

When the athlete overtakes her opponent, they will have covered the same distance.

$$0.4T^2 = 0.3(T+2)^2$$
$$4T^2 = 3(T+2)^2$$
$$4T^2 = 3T^2 + 12T + 12$$
$$T^2 - 12T - 12 = 0$$
$$(T-6)^2 - 36 - 12 = 0$$
$$(T-6)^2 = 48$$
$$T = 6 \pm 4\sqrt{3}$$

Since $T > 0$, $T = 6 + 4\sqrt{3} = 12.9$ s (1 decimal place)

b When $T = 6 + 4\sqrt{3}$, $s = 0.4 \times (6 + 4\sqrt{3})^2 = 66.9$ m

$100 - 66.9 = 33.1$ m

5 a $u = U\,\text{m s}^{-1}$, $a = -9.8\,\text{m s}^{-2}$, $s = 40\,\text{m}$, $v = 0\,\text{m s}^{-1}$

$$v^2 = u^2 + 2as$$
$$0 = U^2 + 2 \times -9.8 \times 40$$
$$U^2 = 784$$
$$U = 28$$

b $u = 28\,\text{m s}^{-1}$, $a = -9.8\,\text{m s}^{-2}$, $s = -14\,\text{m}$

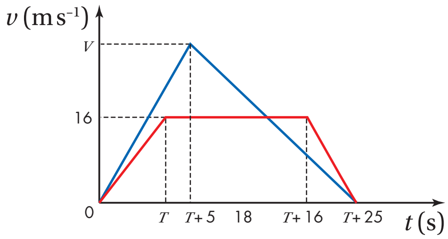
$$v^2 = u^2 + 2as$$
$$v^2 = 28^2 + 2 \times -9.8 \times -14$$
$$v^2 = 1058.4$$
$$v = -32.5$$

Speed $= 32.5\,\text{m s}^{-1}$

c $u = 28\,\text{m s}^{-1}$, $a = -9.8\,\text{m s}^{-2}$, $s = -24\,\text{m}$

$$v^2 = u^2 + 2as$$
$$v^2 = 28^2 + 2 \times -9.8 \times -24$$
$$v^2 = 1254.4$$
$$v = -35.4$$
$$v = u + at$$
$$-35.4 = 28 - 9.8t$$
$$t = 6.47\,\text{s}$$

6 a



b Distance travelled by first motorist

$$= \tfrac{1}{2}(T + 25 + 16) \times 16$$
$$= 8(T + 41)$$

For second motorist, $a = \dfrac{v}{t}$

$$2 = \frac{v}{T+5}$$

So the maximum velocity of second motorist

$$= 2(T+5)$$

Distance travelled by second motorist

$$= \tfrac{1}{2}(T+25) \times 2(T+5)$$
$$8(T+41) = \tfrac{1}{2}(T+25) \times 2(T+5)$$
$$8T + 328 = (T+25)(T+5)$$
$$8T + 328 = T^2 + 30T + 125$$
$$T^2 + 22T - 203 = 0$$
$$(T+29)(T-7) = 0$$
$$T = 7\,\text{s}$$

c $8(7 + 41) = 384$ m

7 **a** For PQ, $t = 4$ s, $s = 96$ m

$s = ut + \frac{1}{2}at^2$

$96 = 4u + 8a$ ①

For PS, $t = 18$ s, $s = 243$ m

$243 = 18u + 162a$

$27 = 2u + 18a$

$54 = 4u + 36a$ ②

① − ②

$-42 = 28a$

$a = -1.5\text{ m s}^{-2}$

$u = 27\text{ m s}^{-1}$

$v = u + at$

$v = 27 - 1.5 \times 18 = 0\text{ m s}^{-1}$

b For PR, $a = -1.5\text{ m s}^{-2}$, $u = 27\text{ m s}^{-1}$, $s = 240$ m

$s = ut + \frac{1}{2}at^2$

$240 = 27t + \frac{1}{2} \times -1.5 \times t^2$

$t^2 - 36t + 320 = 0$

$(t - 16)(t - 20) = 0$

$t = 16$ or 20

20 seconds elapse before the particle passes through R for the second time.

8 **a** $a = (24t - 42)\text{ m s}^{-2}$

Integrating with respect to t: $v = 12t^2 - 42t + C_1$

When $t = 0$, $v = 30\text{ m s}^{-1}$

$C_1 = 30$

$v = 12t^2 - 42t + 30$

Integrating with respect to t: $r = 4t^3 - 21t^2 + 30t + C_2$

When $t = 0$, $r = 3$ m

$C_2 = 3$

$r = 4t^3 - 21t^2 + 30t + 3$

b $v = 12t^2 - 42t + 30 = 0$

$2t^2 - 7t + 5 = 0$

$(2t - 5)(t - 1) = 0$

$t = 1$ s or $\frac{5}{2}$ s

c When $t = 2$, $r = 4(2)^3 - 21(2)^2 + 30(2) + 3 = 11$

When $t = 2.5$, $r = 4(2.5)^3 - 21(2.5)^2 + 30(2.5) + 3 = 9.25$

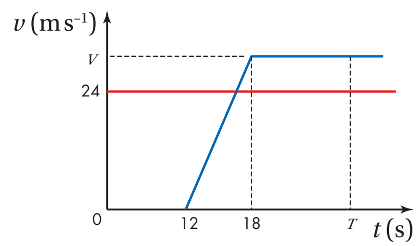
When $t = 3$, $r = 4(3)^3 - 21(3)^2 + 30(3) + 3 = 12$

Distance $= (11 - 9.25) + (12 - 9.25) = 4\frac{1}{2}$ m

9 **a** $24\text{ m s}^{-1} = 86\,400$ metres per hour $= 86.4\text{ km h}^{-1}$

≈ 54 mph, which is faster than the speed limit of 40 mph.

b



Distance travelled by speeding car = $24T$

Distance travelled by police

$= \frac{1}{2}(T - 12 + T - 18) \times V = V(T - 15)$

Since the distance travelled is the same,

$24T = V(T - 15)$

$24T = VT - 15V$

$VT = 24T + 15V$

Since the police travel 1.344 km at $V\text{ m s}^{-1}$,

$V(T - 18) = 1344$

$VT = 18V + 1344$

$24T + 15V = 18V + 1344$

$24T = 3V + 1344$

$24T = 3\left(\frac{1344}{T-18}\right) + 1344$

$24T(T - 18) = 3$ ´ $1344 + 1344(T - 18)$

$24T^2 - 432T = 4032 + 1344T - 24\,192$

$24T^2 - 1776T + 20\,160 = 0$

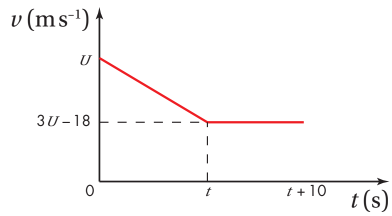
$T^2 - 74T + 840 = 0$

$(T - 14)(T - 60) = 0$

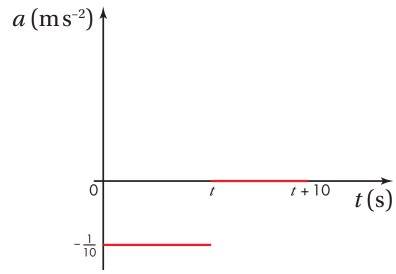
Since $T > 18$, $T = 60$

c $V = \frac{1344}{60-18} = 32$

10 **a** **i**



ii



b Whilst decelerating, $a = -0.1\text{ m s}^{-2}$, $u = U\text{ m s}^{-1}$,

$v = (3U - 18)\text{ m s}^{-1}$

$v^2 = u^2 + 2as$

$s = \frac{v^2 - u^2}{2a}$

$s = \frac{(3U-18)^2 - U^2}{2 \times -0.1}$

Whilst travelling at $(3U - 18)\text{ m s}^{-1}$, $t = 10$ s

$s = 10(3U - 18)$

Total distance = 200

$\frac{(3U-18)^2 - U^2}{2 \times -0.1} + 10(3U - 18) = 200$

$(3U - 18)^2 - U^2 - 2(3U - 18) = -40$

$9U^2 - 108U + 324 - U^2 - 6U + 36 = -40$

$8U^2 - 114U + 400 = 0$

$4U^2 - 57U + 200 = 0$

$(4U - 25)(U - 8) = 0$

$U = 6.25$ or 8

Since $U < 7$, $U = 6.25$

11 a $v = k(16t - t^2)$

Integrating with respect to t: $r = k[8t^2 - \frac{1}{3}t^3]$

$k(8(8)^2 - \frac{1}{3}(8)^3) = 256$

$k(512 - \frac{512}{3}) = 256$

$k = \frac{3}{4}$

b Maximum velocity is at $t = 8$ s.

$v = \frac{3}{4}(8)(16 - 8) = 48\,\text{m s}^{-1}$

c $\frac{1}{2} \times 48 \times (X - 8) = 256$

$X - 8 = \frac{32}{3}$

$X = \frac{56}{3}$

Chapter 16 Forces

1 $W = mg$

$m = \frac{83.3}{9.8} = 8.5\,\text{kg}$

Resolving horizontally: $40 - 23 = 17 = 8.5a$

$a = 2\,\text{m s}^{-2}$

$u = 0\,\text{m s}^{-1}$, $a = 2\,\text{m s}^{-2}$, $s = 25\,\text{m}$

$v^2 = u^2 + 2as$

$v^2 = 0 + 2 \times 2 \times 25 = 100$

$v = 10\,\text{m s}^{-1}$

2 a $u = (-3\mathbf{i} + 5\mathbf{j})\,\text{m s}^{-1}$, $t = 6$ s, $r = (18\mathbf{i} + 3\mathbf{j})$m

$r = ut + \frac{1}{2}at^2$

$(18\mathbf{i} + 3\mathbf{j}) = (-3\mathbf{i} + 5\mathbf{j}) \times 6 + \frac{1}{2}a \times 6^2$

$(18\mathbf{i} + 3\mathbf{j}) = -18\mathbf{i} + 30\mathbf{j} + 18a$

$18a = 36\mathbf{i} - 27\mathbf{j}$

$a = (2\mathbf{i} - 1.5\mathbf{j})\,\text{m s}^{-2}$

$r = ut + \frac{1}{2}at^2$

$r = (-3\mathbf{i} + 5\mathbf{j}) \times 10 + \frac{1}{2}(2\mathbf{i} - 1.5\mathbf{j}) \times 10^2$

$= -30\mathbf{i} + 50\mathbf{j} + 100\mathbf{i} - 75\mathbf{j} = (70\mathbf{i} - 25\mathbf{j})$ m

Distance $= \sqrt{70^2 + (-25)^2} = 74.3$ m

b $\mathbf{F} = m\mathbf{a}$

$\mathbf{F} = 7(2\mathbf{i} - 1.5\mathbf{j}) = (14\mathbf{i} - 10.5\mathbf{j})$ N

Magnitude of $\mathbf{F} = \sqrt{14^2 + (-10.5)^2} = 17.5$ N

c $v = u + at$

$v = (-3\mathbf{i} + 5\mathbf{j}) + (2\mathbf{i} - 1.5\mathbf{j})t$

$= (2t - 3)\mathbf{i} + (5 - 1.5t)\mathbf{j}$

When P is moving parallel to **j**,

$2t - 3 = 0$

$t = 1.5$ s

3 a $\sqrt{(4a)^2 + (-3a)^2} = \pm 5a = 15$

$a = 3$ (since $a > 0$)

b $12\mathbf{i} - 9\mathbf{j} + 3\mathbf{i} - 7\mathbf{j} + b\mathbf{i} + 2\mathbf{j} = 6\mathbf{i} + c\mathbf{j}$

Equating coefficients of **i**: $12 + 3 + b = 6$

$b = -9$

c Equating coefficients of **j**: $-9 - 7 + 2 = c$

$c = -14$

d $\tan^{-1}\left(\frac{14}{6}\right) = 66.8°$

4 a $\mathbf{v} = \mathbf{u} + \mathbf{a}t$

$\mathbf{a} = \frac{v - u}{t}$

$\mathbf{a} = \frac{\begin{bmatrix} 168 \\ 116 \end{bmatrix} - \begin{bmatrix} -48 \\ 311 \end{bmatrix}}{10} = \begin{bmatrix} 21.6 \\ -19.5 \end{bmatrix}$

$\mathbf{F} = m\mathbf{a}$

$\mathbf{F} = \frac{1}{3} \times \begin{bmatrix} 21.6 \\ -19.5 \end{bmatrix} = \begin{bmatrix} 7.2 \\ -6.5 \end{bmatrix}$

Magnitude of $\mathbf{F} = \sqrt{(7.2)^2 + (-6.5)^2} = 9.7$ N

b $\tan^{-1}\left(\frac{6.5}{7.2}\right) = 42°$

Bearing $= 90 + 42 = 132°$

c $\mathbf{r} = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$

$\mathbf{r} = \begin{bmatrix} -48 \\ 311 \end{bmatrix} \times 10 + \frac{1}{2}\begin{bmatrix} 21.6 \\ -19.5 \end{bmatrix} \times 10^2$

$= \begin{bmatrix} -480 \\ 3110 \end{bmatrix} + \begin{bmatrix} 1080 \\ -975 \end{bmatrix} = \begin{bmatrix} 600 \\ 2135 \end{bmatrix}$

Distance $= \sqrt{600^2 + 2135^2} = 2220$ m (3 s.f.)

5 a Tensions are equal.

b Let mass of A = M kg and mass of B = $2M$ kg

Resolving vertically downwards for B:

$2Mg - T = 2Ma$

Resolving vertically upwards for A: $T - Mg = Ma$

Adding: $Mg = 3Ma$

$a = \frac{1}{3}g$

$u = 0\,\text{m s}^{-1}$, $a = \frac{1}{3}g\,\text{m s}^{-2}$, $s = 0.588$ m

$s = ut + \frac{1}{2}at^2$

$0.588 = 0 + \frac{1}{2} \times \frac{1}{3}g \times t^2$

$t^2 = \frac{0.588 \times 6}{g} = 0.36$

$t = 0.6$ s

6 a Resolving horizontally for P: $T - 2.5g = 4a$

Resolving vertically downwards for Q: $6g - T = 6a$

Adding: $3.5g = 10a$

$a = 3.43\,\text{m s}^{-2}$

$u = 0\,\text{m s}^{-1}$, $a = 0.343\,\text{m s}^{-2}$, $s = 3$ m

$v^2 = u^2 + 2as$

$v^2 = 0 + 2 \times 0.343 \times 3 = 2.058$

$v = 1.43\,\text{m s}^{-1}$

b Resolving horizontally for P: $-2.5g = 4a$

$a = -6.125\,\text{m s}^{-2}$

$u = 1.43\,\text{m s}^{-1}$, $a = -6.125\,\text{m s}^{-2}$, $v = 0\,\text{m s}^{-1}$

$v^2 = u^2 + 2as$

$s = \frac{v^2 - u^2}{2a}$

$s = \frac{0 - 2.058}{2 \times -6.125} = 0.168$ m

Since the pulley is 3 m away when Q reaches the ground, P does not reach the pulley.

7 Let the tension with the man in the lift be T_1 N and the tension with the man and boy in the lift be T_2 N.
Resolving upwards with man in lift:
$T_1 - (75 + M)g = (75 + M)a = (75 + M) \times 0.6$
$T_1 = (75 + M) \times 0.6 + (75 + M)g = 10.4(75 + M)$
Resolving upwards with man and boy in lift:
$T_2 - (100 + M)g = (100 + M) \times 0.5$
$T_2 = (100 + M) \times 0.5 + (100 + M)g$
$= 10.3(100 + M)$
$T_2 = T_1 + 215$
$10.3(100 + M) = 10.4(75 + M) + 215$
$1030 + 10.3M = 780 + 10.4M + 215$
$0.1M = 35$
$M = 350$

8 **a** $u = 0\,\text{m s}^{-1}$, $t = 4\,\text{s}$, $v = 6\,\text{m s}^{-1}$
$v = u + at$
$a = \frac{v-u}{t}$
$a = \frac{6-0}{4} = 1.5\,\text{m s}^{-2}$
Let mass of A = $(M + 4)$ kg and mass of B = M kg
Resolving vertically upwards for A:
$113 - T - (M + 4)g = (M + 4)a = (M + 4) \times 1.5$
Resolving vertically upwards for B: $T - Mg = M \times 1.5$
Adding:
$113 - (2M + 4)g = (2M + 4) \times 1.5$
$113 = 11.3(2M + 4)$
$2M + 4 = 10$
$M = 3$
Mass of particle A = 3 + 4 = 7 kg

b Resolving vertically upwards for A:
$113 - 7g = 7a$
$a = 6.34\,\text{m s}^{-2}$

c Whilst connected, $u = 0\,\text{m s}^{-1}$, $t = 4$ s, $a = 1.5\,\text{m s}^{-2}$
$s = ut + \frac{1}{2}at^2$
$s = 0 + \frac{1}{2} \times 1.5 \times 4^2 = 12\,\text{m}$
Whilst unconnected, $u = 6\,\text{m s}^{-1}$, $s = -14\,\text{m}$,
$a = -9.8\,\text{m s}^{-2}$
$v^2 = 6^2 + 2 \times -9.8 \times -14 = 310.4$
$v = 17.6\,\text{m s}^{-1}$
$v = u + at$
$t = \frac{v-u}{a}$
$t = \frac{-17.6-6}{-9.8} = 2.41\,\text{s}$

9 **a** $u = 0\,\text{m s}^{-1}$, $a = 9.8\,\text{m s}^{-2}$, $s = 30\,\text{m}$
$v^2 = u^2 + 2as$
$v^2 = 0 + 2 \times 9.8 \times 30 = 588$
$v = 24.2\,\text{m s}^{-1}$
Resolving vertically downwards when rock hits mud: $12g - 2500 = 12a$
$a = -198.53\,\text{m s}^{-2}$
$v = 0\,\text{m s}^{-1}$, $u = 24.2\,\text{m s}^{-1}$, $a = -198.53\,\text{m s}^{-2}$
$v^2 = u^2 + 2as$
$s = \frac{v^2-u^2}{2a}$
$s = \frac{0-588}{2\times-198.53} = 1.48\,\text{m}$

b Whilst in the air, $u = 0\,\text{m s}^{-1}$, $a = 9.8\,\text{m s}^{-2}$,
$v = 24.2\,\text{m s}^{-1}$
$v = u + at$
$t = \frac{v-u}{a}$
$t = \frac{24.2-0}{9.8} = 2.4743\,\text{s}$
Whilst in the mud, $v = 0\,\text{m s}^{-1}$, $a = -198.53\,\text{m s}^{-2}$,
$u = 24.2\,\text{m s}^{-1}$
$v = u + at$
$t = \frac{v-u}{a}$
$t = \frac{0-24.2}{-198.53} = 0.122\,\text{s}$
Total time = 2.4743 + 0.122 = 2.60 s

10 Resolving horizontally for the van:
$1860 - T - 420 = 500a$
Resolving horizontally for the trailer:
$T - 600 = 900a$
Adding: $840 = 1400a$
$a = 0.6\,\text{m s}^{-2}$
$u = 0\,\text{m s}^{-1}$, $s = 67.5\,\text{m}$, $a = 0.6\,\text{m s}^{-2}$
$v^2 = u^2 + 2as$
$v^2 = 0 + 2 \times 0.6 \times 67.5 = 81$
$v = 9\,\text{m s}^{-1}$
Resolving horizontally for the trailer when the cable snaps: $-600 = 900a$
$a = -\frac{2}{3}\,\text{m s}^{-2}$
$u = 9\,\text{m s}^{-1}$, $v = 0\,\text{m s}^{-1}$, $a = -\frac{2}{3}\,\text{m s}^{-2}$
$v^2 = u^2 + 2as$
$s = \frac{v^2-u^2}{2a}$
$s = \frac{0-81}{2\times-\frac{2}{3}} = 60.75\,\text{m}$
Distance is 60.8 m (3 s.f.)