Collins

A-level Mathematics

Year 2
Student Book

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ebook included

CONTENTS

* Short answers are given in this book, with full worked solutions for all exercises, large data set activities, exam-style questions and extension questions available to teachers by emailing education@harpercollins.co.uk

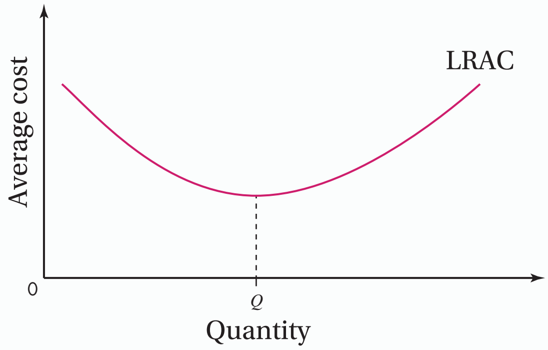
6 DIFFERENTIATION

A business involved in making and selling particular products will have different types of costs. Fixed costs include things such as the rent of business premises and the cost of machinery and vehicles. Marginal costs include the cost of the raw materials used in manufacturing a product or the fuel used to produce it.

Profit will vary with the number of products sold and that in turn is affected by many things, particularly the selling price and the profit margin on each item.

Economists use mathematical models and equations to analyse the connections between these variables, and this can help to make sure that the company makes a profit and is able to plan for expansion.

This graph shows how the long-run average cost (LRAC) varies with the quantity of a product that is produced.



By differentiating the equation for this curve you can determine how varying the quantity produced will affect costs.

The minimum point (where the gradient is zero) shows the quantity Q that will give the lowest possible average cost.

LEARNING OBJECTIVES

You will learn how to:

- identify all the types of stationary point on a curve
- use the chain rule to differentiate a range of functions
- differentiate exponential and logarithmic functions
- differentiate sine and cosine functions.

TOPIC LINKS

For this chapter you should be able to differentiate linear combinations of powers of a variable. From **Book 1, Chapter 7 Exponentials and logarithms** you should be familiar with exponential functions and their graphs, including those involving the constant e, and be able to manipulate functions involving logarithms to base e. From **Chapter 5 Trigonometry** you should know about using radians to measure angles and small angle approximations to sines and cosines.

PRIOR KNOWLEDGE

You should already know how to:

- identify maximum and minimum points on a graph
- work out any power of a variable, including fractional and negative indices
- sketch graphs of $y = e^{kx}$ and $y = \ln x$
- use radian measure for angles
- use small angle approximations for $\sin\theta$ and $\cos\theta$.

You should be able to complete the following questions correctly:

1 Find the stationary points of the graph of $y = x^3 - 6x^2$ and identify whether they are maximum or minimum points.

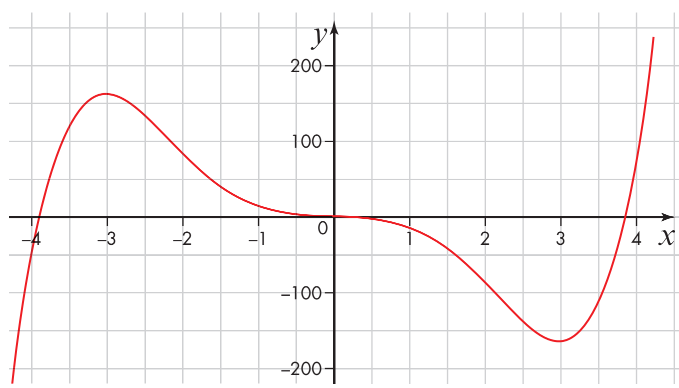
2 Show that if α is small then $\cos 2\alpha \approx 1 - 2\alpha^2$.

3 Differentiate with respect to x:

a $2x(x^2 - 1)$ **b** $\frac{x^2 - 1}{2x}$ **c** $\frac{1}{\sqrt{x}}$

6.1 Turning points

This is a graph of $y = x^5 - 15x^3$



At a **stationary point**, the gradient of the curve is zero; that is, $\frac{dy}{dx} = 0$.

To find the stationary points, differentiate the equation.

$$\frac{dy}{dx} = 5x^4 - 45x^2$$

When $\frac{dy}{dx} = 0$, $5x^4 - 45x^2 = 0$.

Divide by 5 and factorise.

$$x^4 - 9x^2 = 0$$

$$x^2(x^2 - 9) = 0$$

$$x^2(x - 3)(x + 3) = 0 \text{ so } x = 0, 3 \text{ or } -3$$

You can see on the graph that at these points the gradient is 0.

These stationary points are (0, 0), (3, –162) and (–3, 162).

(–3, 162) is a **maximum point**. The y-coordinate at this point is less than the y-coordinates at nearby points. The gradient of the curve changes sign from positive to negative as you go past that point.

(3, –162) is a **minimum point**. The y-coordinate at this point is less than the y-coordinates at nearby points. At this point, the gradient changes from negative to positive.

(0, 0) is a **point of inflection**. The gradient at neighbouring points on either side is negative.

A section of a curve that has increasing gradient (positive second derivative) is known as concave upwards, and a section of a curve that has decreasing gradient (negative second derivative) is known as concave downwards.

An alternative to judging the type of stationary point by eye is to use the **second derivative**.

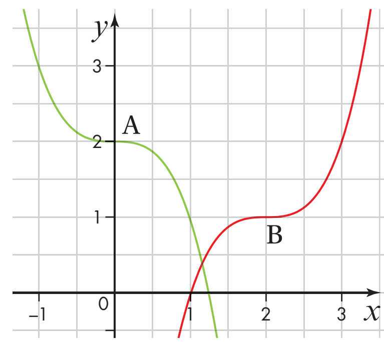
$$\frac{d^2y}{dx^2} = 20x^3 - 90x$$

When $x = -3$, $\frac{d^2y}{dx^2} = -270 < 0$, which confirms this is a maximum point.

When $x = 3$, $\frac{d^2y}{dx^2} = 270 > 0$, which confirms this is a minimum point.

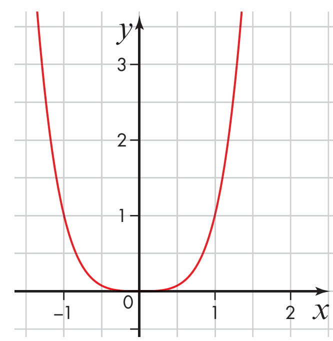
When $x = 0$, $\frac{d^2y}{dx^2} = 0$. At a point of inflection the second derivative is always 0. However, this is not conclusive and the point could be a maximum, a minimum or a point of inflection. Look at the value of the gradient on either side of the point to decide which it is.

At the point of inflection, the curve changes from concave upwards to concave downwards (or vice versa), so the second derivative at this point is zero.



There are two types of stationary points that are points of inflection, illustrated by A and B on the upper graph.

The lower graph shows the curve $y = x^4$.



The origin is clearly a minimum point but $\frac{d^2y}{dx^2} = 12x^2$, which is 0 when $x = 0$.

Here is a summary:

Value of $\frac{d^2y}{dx^2}$	negative	positive	zero
Type of stationary point	maximum point	minimum point	cannot tell

Exercise 6.1A

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1 Find the stationary point of the graph of $y = 16 + 10x - x^2$ and state, with a reason, whether it is a maximum or a minimum point.

2 **a** Show that the graph of $y = x^3 - 3x^2 + 3x - 20$ has just one stationary point.

b Find its coordinates and show that it is a point of inflection.

3 Find and describe the stationary points for the graph of $f(x) = x^4 - 2x^2$.

4 **a** Show that the graph of $y = x^3$ has a point of inflection at the origin.

b Describe any stationary points for the curve $y = (x - 10)^3 + 20$. Justify your answer.

PF 5 $y = ax^2 + bx + c$, $a \neq 0$, is a curve.

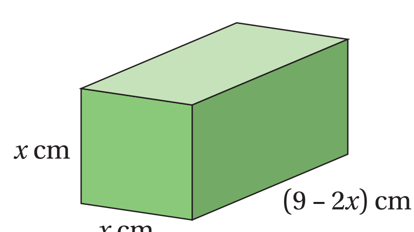
Show that it must have a stationary point, which can be a maximum or a minimum but not a point of inflection.

PS PF 6 The sides of a cuboid are x cm, x cm and $(9 - 2x)$ cm.

a Show that the total length of all the edges is 36 cm.

b Show that the volume is a maximum when the shape is a cube.

c Show that the total surface area is also a maximum when the shape is a cube.



6.2 The chain rule

If you need to differentiate a function such as $y = (2x+3)^3$ and find $\frac{dy}{dx}$, one way is to multiply out the bracket and then differentiate each term. That will take a lot of work. Is there a better way?

You can break the function into two parts by writing $y = u^3$ where $u = 2x + 3$.

Both of these are easy to differentiate: $\frac{dy}{du} = 3u^2$ and $\frac{du}{dx} = 2$.

How does this help? Remember that to differentiate from first principles you increase the variable by a small amount.

Suppose δx is a small increase in x – then both u and y will change. There will be a small change, δu, in u, and a small change, δy, in y.

The derivatives are $\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$, $\frac{dy}{du} = \lim_{\delta u \to 0} \frac{\delta y}{\delta u}$ and $\frac{du}{dx} = \lim_{\delta x \to 0} \frac{\delta u}{\delta x}$

Now we can write $\frac{\delta y}{\delta u} = \frac{\delta y}{\delta u} \times \frac{\delta u}{\delta x}$

This is just the ordinary rule for multiplying fractions, since δx, δu and δy are just numbers and the δu cancels.

Then $$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{\delta y}{\delta u} \times \frac{\delta u}{\delta x} = \lim_{\delta u \to 0} \frac{\delta y}{\delta u} \times \lim_{\delta x \to 0} \frac{\delta u}{\delta x}$$

So $$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

Using this formula in our example:

$$\frac{dy}{du} = 3u^2 \quad \text{and} \quad \frac{du}{dx} = 2 \quad \text{so} \quad \frac{dy}{dx} = 3u^2 \times 2 = 6u^2$$

The final step is to replace u with $2x + 3$, which gives

$$\frac{dy}{dx} = 6(2x+3)^2$$

With practice you can use this method without actually writing down u.

Think of it is as $y = (\text{bracket})^3$ where the bracket stands for $2x + 3$.

Think of it as:	Write it as:
$y = (\text{bracket})^3$	$y = (2x+3)^3$
$\frac{dy}{dx} = \frac{dy}{d(\text{bracket})} \times \frac{d(\text{bracket})}{dx}$	$\frac{dy}{dx} = 3(2x+3)^2 \times 2$ $= 6(2x+3)^2$

This method is called the **chain rule** and it greatly increases the number of functions that you can differentiate.

KEY INFORMATION

If $y = f(u)$ and $u = g(x)$ then $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

Example 1

Differentiate $y = \frac{4}{x^2+1}$.

Solution

Method 1

Write it as $y = 4(x^2+1)^{-1} = 4u^{-1}$ where $u = x^2 + 1$.

Then $\frac{dy}{du} = 4 \times -1 \times u^{-2} = -\frac{4}{u^2}$ and $\frac{du}{dx} = 2x$.

Multiply these.

$$\frac{dy}{dx} = -\frac{4}{u^2} \times 2x = -\frac{8x}{u^2} = -\frac{8x}{(x^2+1)^2}$$

Method 2

If you do not use u explicitly, think of it as $y = 4(\text{bracket})^{-1}$

so that $\frac{dy}{dx} = 4 \times -1 \times (\text{bracket})^{-2} \times \frac{d(\text{bracket})}{dx}$

and write $\frac{dy}{dx} = -4(x^2+1)^{-2} \times 2x = \frac{-8x}{(x^2+1)^2}$ as before.

Use whichever method you are more comfortable with.

You will probably want to introduce u at first and you should find you can do without it as you become more confident.

Stop and think Would you have been able to differentiate the function in **Example 1** if you had not known about the chain rule?

Exercise 6.2A

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1 Find $\frac{dy}{dx}$ in the following cases:

a $y = (4x+2)^2$ **b** $y = (4x+2)^3$ **c** $y = (4x+2)^5$

2 Differentiate with respect to x:

a $(8-x)^{10}$ **b** $(1+3x^2)^{10}$ **c** $(6x-3x^2)^{10}$

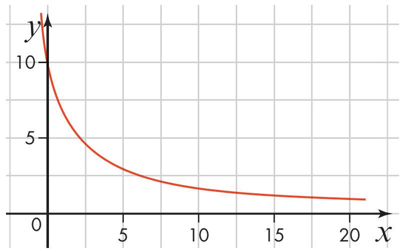
3 Find $f'(x)$ in the following cases:

a $f(x) = \sqrt{x-3}$ **b** $f(x) = \sqrt{2x-3}$ **c** $f(x) = \sqrt{x^2-3}$

4 The equation of this curve is $y = \frac{20}{2+x}$, $x > 0$.

a Find the gradient at (2, 5).

b Find the coordinates of the point where the gradient is –0.2.

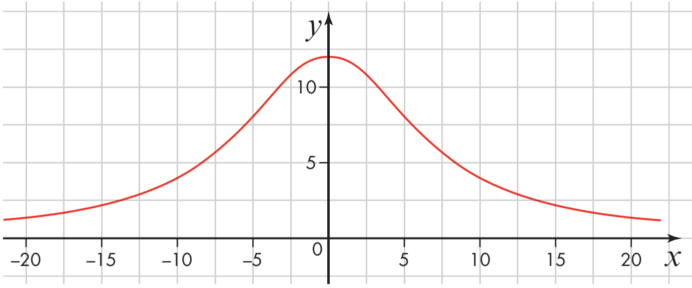


5 The equation of this curve is $y = \frac{600}{x^2+50}$.

a Show that the point (10, 4) is on the curve.

b Find $\frac{dy}{dx}$.

c Find the equation of the tangent at (10, 4).



6.3 Differentiating e^{kx}

Here is a proof that if $y = e^x$ then $\frac{dy}{dx} = e^x$.

Suppose x increases by a small amount, δx, and y changes by $\delta y = e^{x+\delta x} - e^x$.

$$\frac{\delta y}{\delta x} = \frac{e^{x+\delta x} - e^x}{\delta x} = \frac{e^x(e^{\delta x} - 1)}{\delta x}$$

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{e^x(e^{\delta x} - 1)}{\delta x} = e^x \lim_{\delta x \to 0} \frac{e^{\delta x} - 1}{\delta x}$$

So what is $\lim_{\delta x \to 0} \frac{e^{\delta x} - 1}{\delta x}$?

Here is a table of values, rounded to 4 decimal places:

δx	0.1	0.01	0.001
$\frac{e^{\delta x} - 1}{\delta x}$	1.0517	1.0050	1.0005

It is reasonable to assume that $\lim_{\delta x \to 0} \frac{e^{\delta x} - 1}{\delta x} = 1$.

Therefore $\frac{dy}{dx} = e^{\delta x} \lim_{\delta x \to 0} \frac{e^{\delta x} - 1}{\delta x} = e^x$

This means that for any point on the graph of $y = e^x$, the gradient is just the y-coordinate.

If $y = e^{kx}$ where k is a constant, then you can find $\frac{dy}{dx}$ by using the chain rule.

If $u = kx$ and $y = e^u$ then

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = e^u \times k = ke^{kx}$$

If you multiply the function by a constant, the derivative is multiplied by the constant in the usual way.

If $y = ce^{kx}$, then $\frac{dy}{dx} = cke^{kx}$.

PROOF

Factorising the expression for $\frac{\delta y}{\delta x}$ leads you to consider a simpler limit.

KEY INFORMATION

If $y = e^x$ then $\frac{dy}{dx} = e^x$, or if $f(x) = e^x$ then $f'(x) = e^x$.

KEY INFORMATION

If $y = e^{kx}$ then $\frac{dy}{dx} = ke^{kx}$, or if $f(x) = e^{kx}$ then $f'(x) = ke^{kx}$.

Example 2

Find the gradient of the curve $y = 10e^{-0.3x}$ at the point where $x = 5$.

Solution

If $y = 10e^{-0.3x}$ then $\frac{dy}{dx} = -0.3 \times 10e^{-0.3x} = -3e^{-0.3x}$.

When $x = 5$, $\frac{dy}{dx} = -3e^{-0.3\times 5} = -3e^{-1.5} = -0.669$ to 3 d.p.

Sometimes exponential functions are written in the form a^{kx}, where a is not e but a different number. **Example 3** shows you how to differentiate such an expression.

Example 3

The population of a city, y millions, in x years' time is predicted to be given by the formula

$$y = 5 \times 1.02^x$$

Find the expected rate of increase in 4 years' time.

Solution

You need to find the value of $\frac{dy}{dx}$ when $x = 4$.

If $1.02 = e^a$ then $a = \ln 1.02$ and so $1.02 = e^{\ln 1.02}$.

That means $y = 5 \times (e^{\ln 1.02})^x = 5e^{(\ln 1.02)x}$

Therefore $\frac{dy}{dx} = 5(\ln 1.02)e^{(\ln 1.02)x} = 5(\ln 1.02)1.02^x$

When $x = 4$, $\frac{dy}{dx} = 5(\ln 1.02)1.02^4 = 0.107$ to 3 d.p.

The expected rate of increase is 107 000 to 3 s.f.

This uses the usual rule for combining indices.

The units are millions per year.

Here is the general method.

If you want to differentiate $= a^{kx}$, use the fact that $a = e^{\ln a}$ and write the expression as a power of e.

If $y = a^{kx} = (e^{\ln a})^{kx} = e^{(\ln a)kx}$

then $\frac{dy}{dx} = k(\ln a)e^{(\ln a)kx} = k(\ln a)a^{kx}$

KEY INFORMATION

If $y = x^{kx}$ then $\frac{dy}{dx} = k(\ln a)\, a^{kx}$.

Exercise 6.3A

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1 Find $\frac{dy}{dx}$ in the following cases:

a $y = e^{2x}$ **b** $y = e^{-x}$ **c** $y = e^{0.4x}$ **d** $y = e^{4x+2}$

2 Work out $f'(x)$ in the following cases:

a $f(x) = 4e^{0.5x}$ **b** $f(x) = 100e^{-0.1x}$ **c** $f(x) = 50e^{2x-10}$

3 **a** Write 1.5^x in the form e^{kx} where k is a constant.

b If $y = 1.5^x$, find $\frac{dy}{dx}$.

4 $f(x) = 4e^{2x}$

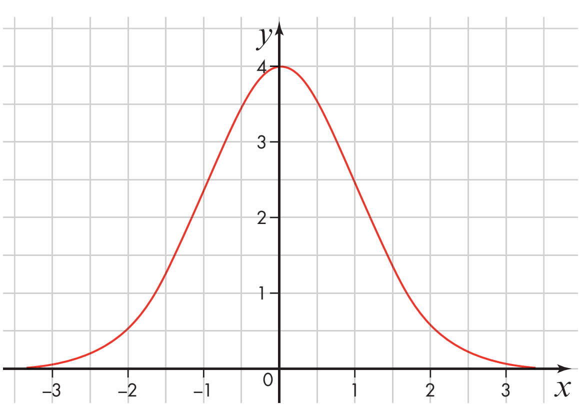
Find:

a $f'(x)$ **b** $f''(x)$

5 The equation of this curve is $y = 4e^{-0.5x^2}$.

a Find $\frac{dy}{dx}$.

b Find the gradient at the point where the x-coordinate is –2.



M **6** A saver invests £5000 at an annual rate of 3% compound interest.

a Show that after t years the value is £5000×1.03^t.

b Write 5000×1.03^t in the form $5000e^{at}$.

c Find the rate at which the savings are increasing when $t = 3$.

PS **7** $f(x) = e^{2x} + e^{-2x}$

Show that $f''(x) = 4f(x)$.

PS **8** The equation of a curve is $y = e^x + 4e^{-x}$.

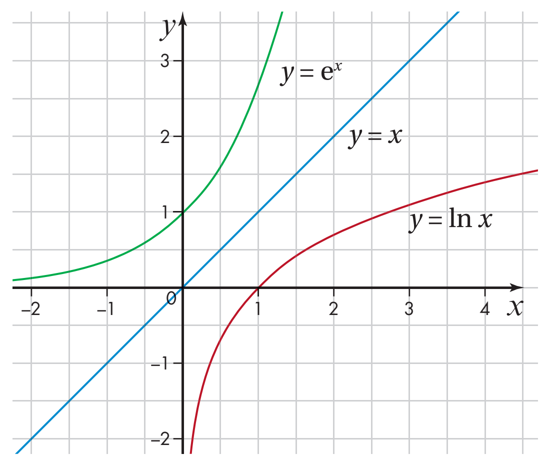
Show that the curve has a minimum point and find its coordinates.

6.4 Differentiating ln *x*

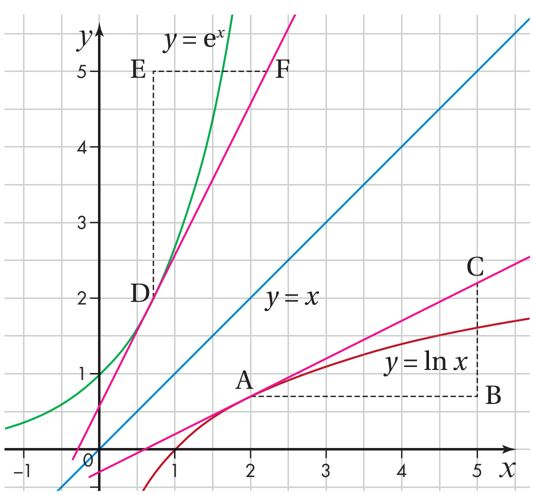
On the right is a graph of $y = \ln x$.

On the same axes is a graph of $y = e^x$.

Because finding e to a power and finding ln are inverse operations, the graph of $y = \ln x$ is a reflection of the graph of $y = e^x$ in the line $y = x$.



Suppose you want to find the gradient of the curve $y = \ln x$ at the point A.



The tangent at A has been drawn; its gradient is $\frac{CB}{AB}$.

DEF is a reflection of ABC in the line $y = x$ so $\frac{CB}{AB} = \frac{FE}{DE}$.

But $\frac{FE}{DE} = \frac{1}{\frac{DE}{FE}} = \frac{1}{y\text{-coordinate of D}}$ because $\frac{DE}{FE}$ is the gradient of $y = e^x$ at the point D, which is just e^x or the y-coordinate.

Because of the reflection, the y-coordinate of D is the x-coordinate of A.

This gives the result that if $y = \ln x$ then $\frac{dy}{dx} = \frac{1}{x}$.

KEY INFORMATION

If $y = \ln x$ then $\frac{dy}{dx} = \frac{1}{x}$

Example 4

Differentiate: **a** $\ln(2x+3)$ **b** $\ln x^2$

Solution

a Use the chain rule:

If $y=\ln(2x+3)$ then $\frac{dy}{dx}=\frac{1}{2x+3}\times 2=\frac{2}{2x+3}$

b Either use the chain rule:

If $y=\ln x^2$ then $\frac{dy}{dx}=\frac{1}{x^2}\times 2x=\frac{2x}{x^2}=\frac{2}{x}$

Or use the properties of logarithms:

$\ln x^2=2\ln x$

So $\frac{dy}{dx}=2\times\frac{1}{x}=\frac{2}{x}$, which is the same result.

Using the fact that $\ln a^k=k\ln a$.

Example 5

Find the equation of the tangent to the curve $y=\ln(x+e)$ at the point (0, 1).

Solution

Use the chain rule to find $\frac{dy}{dx}$.

$y=\ln u$ and $u=x+e$

$\frac{dy}{du}=\frac{1}{u}$ and $\frac{du}{dx}=1$

$$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}=\frac{1}{u}\times 1=\frac{1}{x+e}$$

At (0, 1), $x=0$ so $\frac{dy}{dx}=\frac{1}{e}$

The equation of the tangent is $y-1=\frac{1}{e}(x-0)$

$$y=\frac{x}{e}+1$$

Exercise 6.4A

Answers page 447

1 Differentiate with respect to x:

a $\ln 3x$ **b** $\ln x^3$ **c** $\ln(x^3+2)$

2 Find $\frac{dy}{dx}$ in the following cases:

a $y=4\ln x$ **b** $y=\ln 4x$ **c** $y=\ln x^4$

3 **a** On the same axes, sketch the graphs of $y=\ln x$ and $y=\ln 2x$.

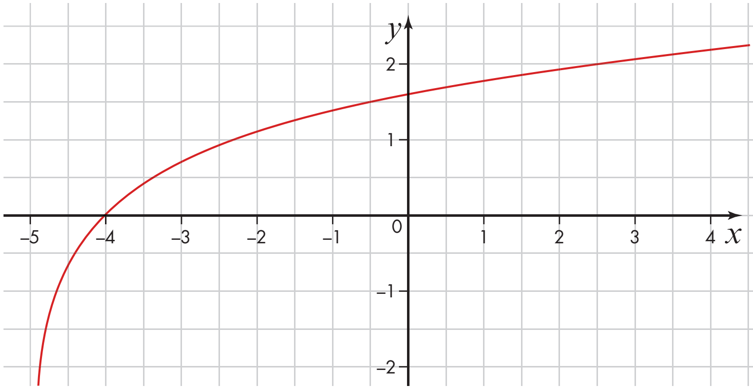
b Find the vector for the translation that maps the graph of $y=\ln x$ onto $y=\ln 2x$.

c $f(x)=\ln kx$ where k is a positive constant, $x>0$.

Show that $f'(x)=\frac{1}{x}$.

d How are your answers to **parts b** and **c** related?

4 This is a graph of the curve $y = \ln(x + 5)$:



a Find the equation of the tangent to the curve at the point where it crosses the x-axis.

b Find the equation of the tangent to the curve at the point where it crosses the y-axis.

PS **5** **a** Show that the point (e, 1) is on the graph of the curve $y = \ln x$.

b Show that the tangent to the curve $y = \ln x$ at (e, 1) passes through the origin.

c Show that the normal to the curve $y = \ln x$ at (e, 1) crosses the x-axis at $e + \frac{1}{e}$.

PF **6** **a** Find $\frac{dy}{dx}$ when $y = \ln\frac{10}{x}$.

b Find $\frac{dy}{dx}$ when $y = \ln\frac{10}{x^2}$.

c Generalise your results of **parts a** and **b** to find $\frac{dy}{dx}$ when $y = \ln\frac{10}{x^n}$ where n is a positive integer.

PS **7** $y = \log_{10} x$

Show that $\frac{dy}{dx} = \frac{1}{x \ln 10}$.

6.5 Differentiating sin *x* and cos *x* from first principles

In this section you will find the derivatives of $\sin x$ and $\cos x$.

Suppose $y = \sin x$.

If δx is a small increase in the value of x and δy is the corresponding change in y, then

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$$

$$\delta y = \sin(x + \delta x) - \sin x$$

so $$\frac{\delta y}{\delta x} = \frac{\sin(x + \delta x) - \sin x}{\delta x}$$

Use the **addition formula**, $\sin(x + \delta x) = \sin x \cos \delta x + \cos x \sin \delta x$.

$$\frac{\delta y}{\delta x} = \frac{\sin x \cos \delta x + \cos x \sin \delta x - \sin x}{\delta x}$$

See **Chapter 5 Trigonometry** for more on the addition formulae.

Factorise the numerator.

$$\frac{\delta y}{\delta x} = \frac{\cos x \sin \delta x + \sin x(\cos \delta x - 1)}{\delta x}$$

Write the expression as two separate fractions.

$$\frac{\delta y}{\delta x} = \cos x\left(\frac{\sin \delta x}{\delta x}\right) + \sin x\left(\frac{\cos \delta x - 1}{\delta x}\right)$$

$\frac{dy}{dx}$ is the limit of this as $\delta x \to 0$.

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0}\left\{\cos x\left(\frac{\sin \delta x}{\delta x}\right) + \sin x\left(\frac{\cos \delta x - 1}{\delta x}\right)\right\}$$

$$= \cos x \lim_{\delta x \to 0} \frac{\sin \delta x}{\delta x} + \sin x \lim_{\delta x \to 0} \frac{\cos \delta x - 1}{\delta x}$$

From **Chapter 5 Trigonometry** you should remember that if θ is small then $\sin\theta \approx \theta$ and $\cos\theta \approx 1 - \frac{1}{2}\theta^2$.

Use these to give the approximations $\sin \delta x \approx \delta x$ and $\cos \delta x \approx 1 - \frac{1}{2}(\delta x)^2$.

$$\frac{\sin \delta x}{\delta x} \approx 1 \quad \text{and} \quad \frac{\cos \delta x - 1}{\delta x} \approx \frac{1}{2}\delta x$$

$$\lim_{\delta x \to 0} \frac{\sin \delta x}{\delta x} = 1 \quad \text{and} \quad \lim_{\delta x \to 0} \frac{\cos \delta x - 1}{\delta x} \approx \lim_{\delta x \to 0} \frac{1}{2}\delta x = 0$$

Hence the result is $\frac{dy}{dx} = \cos x$.

This gives the very simple result that if $y = \sin x$ then $\frac{dy}{dx} = \cos x$.

Stop and think Would this result still be true if the angle was in degrees rather than in radians?

You can differentiate $y = \cos x$ from first principles in a similar way.

$$\frac{\delta y}{\delta x} = \frac{\cos(x + \delta x) - \cos x}{\delta x}$$

$$= \frac{\cos x \cos \delta x - \sin x \sin \delta x - \cos x}{\delta x}$$

$$= \frac{\cos x(\cos \delta x - 1) - \sin x \sin \delta x}{\delta x}$$

$$\frac{dy}{dx} = \cos x \lim_{\delta x \to 0} \frac{\cos \delta x - 1}{\delta x} - \sin x \lim_{\delta x \to 0} \frac{\sin \delta x}{\delta x}$$

Which means that $\frac{dy}{dx} = -\sin x$.

So if $y = \cos x$, $\frac{dy}{dx} = -\sin x$.

KEY INFORMATION

- If $y = \sin x$, then $\frac{dy}{dx} = \cos x$.
- If $y = \cos x$, then $\frac{dy}{dx} = -\sin x$.

Example 6

Differentiate with respect to x:

a $2\sin 4x$ **b** $\cos 3x^2$

Solution

Use the chain rule in both cases.

a $\frac{dy}{dx} = 2\cos 4x \times 4 = 8\cos 4x$

b $\frac{dy}{dx} = -\sin 3x^2 \times 6x = -6x\sin 3x^2$

Exercise 6.5A

Answers page 447

1 Find $\frac{dy}{dx}$ in the following cases:

a $y = \sin 2x$ **b** $y = \cos(5x - 2)$ **c** $y = \sin(x^2 + 1)$

2 Differentiate with respect to x:

a $10\sin 0.5x$ **b** $\sin 3x + \cos 6x$ **c** $\cos(x^2 - 3x - 4)$

3 $f(x) = \sin 4x + \cos 4x$

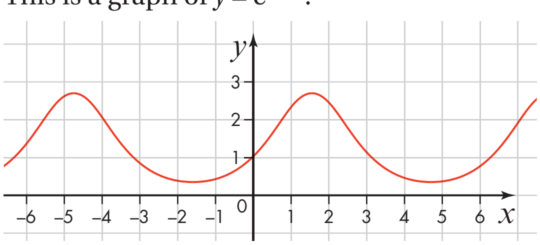
Show that $f''(x) + 16f(x) = 0$

PF **4** **a** Show that the derivative of $\sin^2 x$ is $\sin 2x$.

b Find the derivative of $\cos^2 x$.

c Explain the relationship between your answers to **parts a** and **b**.

5 This is a graph of $y = e^{\sin x}$.



a Find the gradient at the point (0, 1).

b Prove that the graph has a stationary point at $\left(\frac{\pi}{2}, e\right)$.

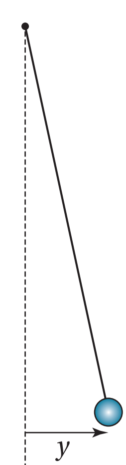
M **6** A bob on the end of a long string is a pendulum making small oscillations.

The displacement, y metres, of the bob after t seconds is given by $y = 0.1\sin 2.4t$.

a Find the speed of the bob when $t = 1$.

b Find the position of the bob when the speed is zero.

c Find the position of the bob when the acceleration is zero.



SUMMARY OF KEY POINTS

- If $f'(x) = 0$ and $f''(x) > 0$ then the curve has a minimum point. If $f'(x) = 0$ and $f''(x) < 0$ then the curve has a maximum point.
- The chain rule: if $y = f(u)$ and $u = g(x)$ then $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$.
- If $y = e^{kx}$ then $\frac{dy}{dx} = ke^{kx}$.
- If $y = \ln x$ then $\frac{dy}{dx} = \frac{1}{x}$.
- If $y = \sin x$ then $\frac{dy}{dx} = \cos x$; if $y = \cos x$ then $\frac{dy}{dx} = -\sin x$.

EXAM-STYLE QUESTIONS 6

Answers page 448

1 The equation of a curve is $y = (x - 10)^3 + 15$.

a Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$. **[3 marks]**

b Show that the curve has a point of inflection and find its coordinates. **[3 marks]**

2 $y = \sqrt{1 + x^2}$

Show that $\frac{dy}{dx} = \frac{x}{y}$. **[4 marks]**

3 $f(x) = \frac{1}{\cos x}$

Show that $f'(x) = \frac{\tan x}{\cos x}$. **[4 marks]**

4 Differentiate with respect to x:

a $y = e^{-2x}$ **[2 marks]**

b $y = e^{-x^2}$ **[2 marks]**

5 The equation of a curve is $y = \ln x^2$, $x > 0$.

Find the coordinates of the point on the curve where the gradient is 0.5. **[5 marks]**

CM **6** Here are attempts by two students to differentiate $\sin(x + 2)$:

Student A	Student B
$y = \sin(x + 2)$	$y = \sin(x + 2)$
$= \sin x + \sin 2$	Use the addition formula.
So	$y = \sin x \cos 2 + \cos x \sin 2$
$\frac{dy}{dx} = \cos x + 0$	So
$= \cos x$	$\frac{dy}{dx} = \cos x \cos 2 - \sin x \sin 2$

Is either student correct? Give a reason for your answer. **[4 marks]**

M **7** The area covered by a colony of bacteria on a flat surface is $10\,\text{mm}^2$.

The area $y\,\text{mm}^2$, in t hours' time is modelled by the formula $y = 10 \times 2^{1.5t}$.

a Show that the area will double in size in 40 minutes. **[2 marks]**

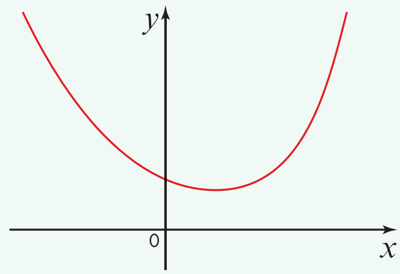
b Find the area in 3 hours' time. **[2 marks]**

c Find the rate at which the area will be increasing in 3 hours' time. **[6 marks]**

d Why might the model no longer be valid after several hours? **[1 mark]**

8 The equation of this curve is $y = 0.1e^x + e^{-0.5x}$.

The curve has a minimum point. Find its coordinates. **[7 marks]**



9 The equation of a curve is $y = 2^{0.5x+3}$.

Find the equation of the tangent to the curve where it crosses the y-axis. **[4 marks]**

CM **10** The equation of a curve is $y = x^3 + 3x^2 + 4x + 5$.

a Show that the curve has no stationary points. **[2 marks]**

b The curve has a point of inflection.

Find the equation of the tangent at this point. **[2 marks]**

11 The equation of a curve is $y = ae^{kx}$ where a and k are constants.

The point (10, 20) is on the curve and the tangent at that point passes through (0, –5) on the y-axis.

Find the values of a and k. **[4 marks]**

PF **12** $y = \sin^3 x$

a Show that $\frac{dy}{dx} = 3\cos x - 3\cos^3 x$. **[4 marks]**

b Show that $\frac{d^2y}{dx^2} = 6\sin x - 9\sin^3 x$. **[4 marks]**

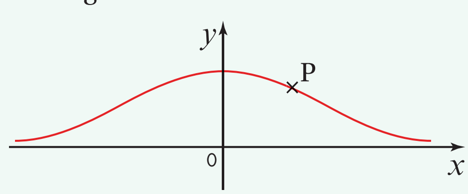
13 The equation of a curve is $y = 10\cos 5(x - 20)°$.

Find the maximum and minimum values of the gradient of this curve. **[4 marks]**

14 The equation of this curve is $y = e^{-\frac{1}{2}x^2}$

The point P has x-coordinate a.

Show that the tangent at P crosses the x-axis at $(a + \frac{1}{a}, 0)$. **[4 marks]**



PF **15** $y = \sin 2x$

Prove from first principles that $\frac{dy}{dx} = 2\cos 2x$. **[5 marks]**