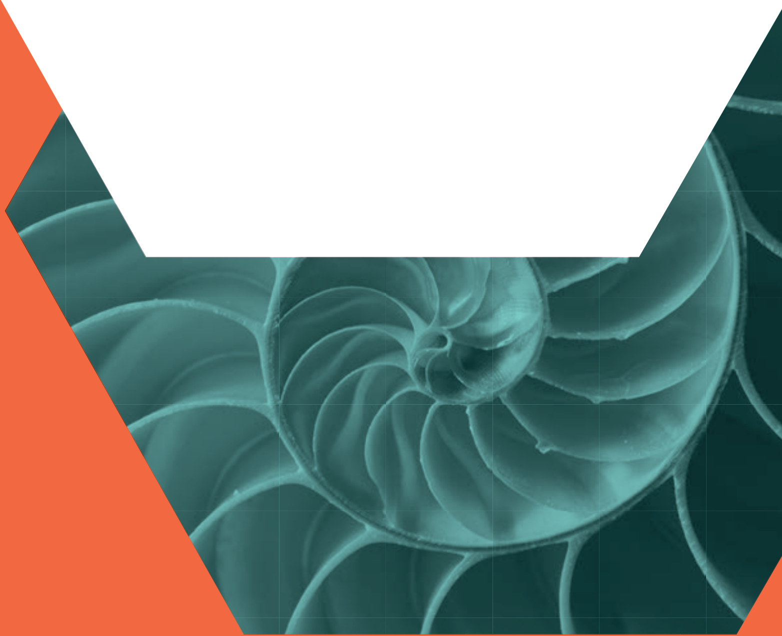
Collins



Maths

4th Edition

GCSE

Higher Student Book

Kevin Evans
Keith Gordon
Brian Speed
Michael Kent

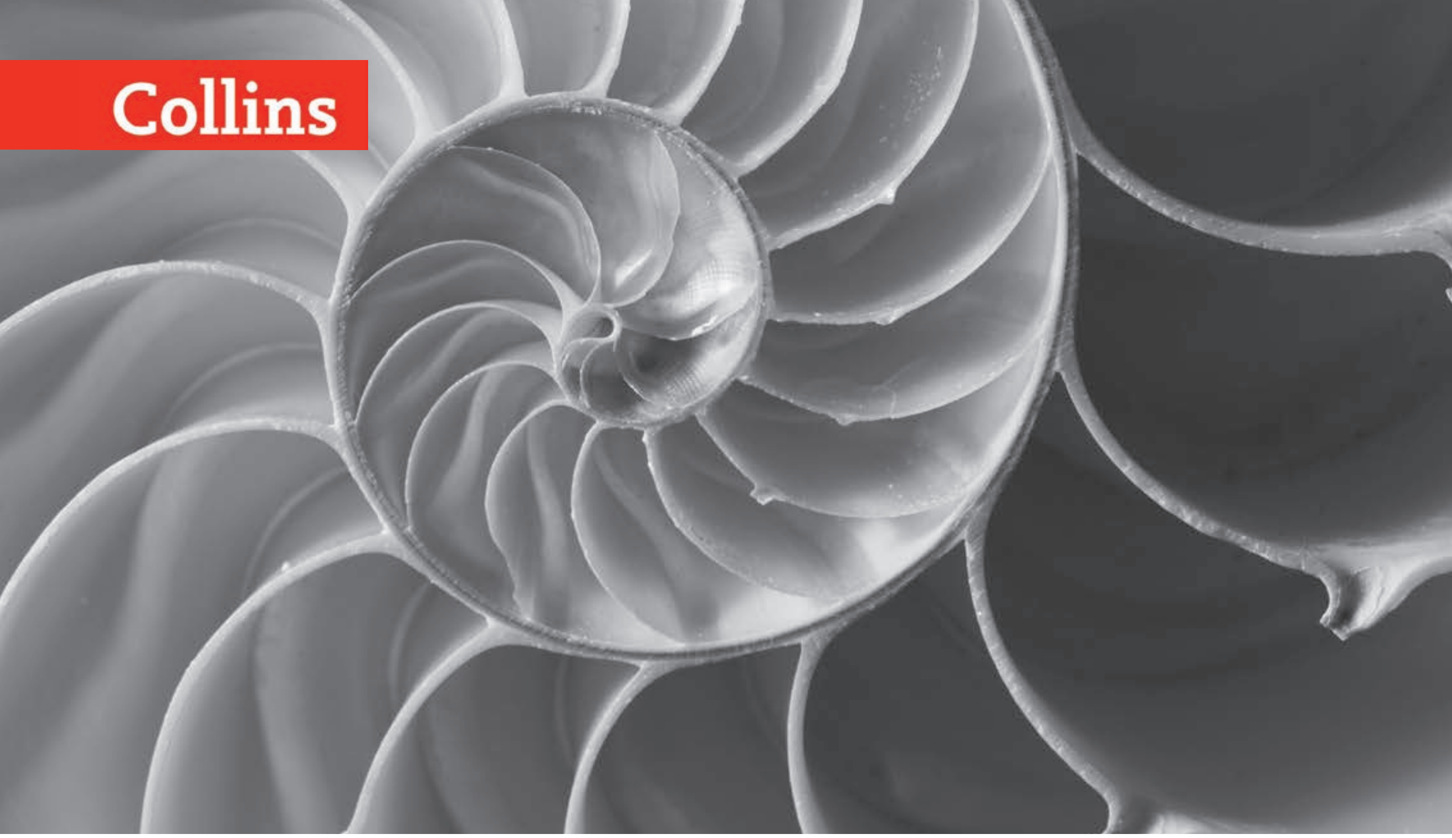
William Collins' dream of knowledge for all began with the publication of his first book in 1819. A self-educated mill worker, he not only enriched millions of lives, but also founded a flourishing publishing house. Today, staying true to this spirit, Collins books are packed with inspiration, innovation and practical expertise. They place you at the centre of a world of possibility and give you exactly what you need to explore it.

Collins. Freedom to teach.

Published by Collins
An imprint of HarperCollins*Publishers*
1 London Bridge Street
London
SE1 9GF

Browse the complete Collins catalogue at
www.collins.co.uk

Cover design by We are Laura
Cover images © Procy/Shutterstock, joingate/Shutterstock
Printed by Fuller Davies www.fullerdavies.com



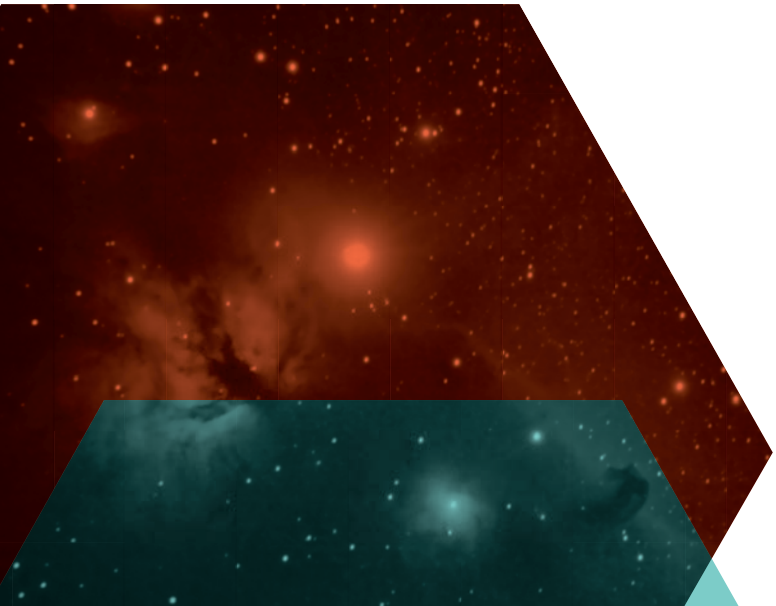
Collins

Maths

4th Edition

GCSE

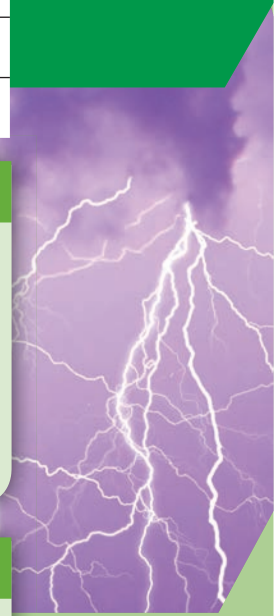
Higher Student Book



Kevin Evans
Keith Gordon
Brian Speed
Michael Kent

24 Algebra: Algebraic fractions and functions

Skill focus key	
MR	mathematical reasoning
CM	communicate mathematically
PS	problem-solving and making connections
EV	evaluate and interpret



This chapter is going to show you:

- how to combine fractions algebraically and solve equations with algebraic fractions
- how to rearrange and change the subject of a formula where the subject appears twice, or as a power
- how to find the inverse function and the composite of two functions
- how to find an approximate solution for an equation using the process of iteration.

You should already know:

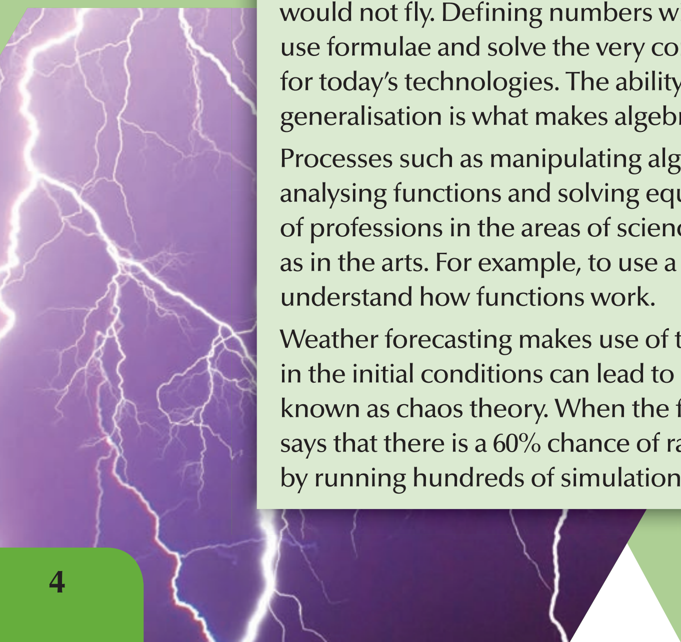
- how to substitute numbers into an algebraic expression
- how to factorise linear and quadratic expressions
- how to expand a pair of linear brackets to get a quadratic equation.

About this chapter

Without algebra, humans would not have reached the moon and aeroplanes would not fly. Defining numbers with letters allows mathematicians to use formulae and solve the very complicated equations that are needed for today's technologies. The ability to move from a special case to a generalisation is what makes algebra so useful.

Processes such as manipulating algebraic fractions, rearranging formulae, analysing functions and solving equations by iteration are used in a variety of professions in the areas of science, engineering and computing as well as in the arts. For example, to use a spreadsheet competently, you need to understand how functions work.

Weather forecasting makes use of the iterative process where small changes in the initial conditions can lead to completely different results. This is known as chaos theory. When the forecaster on television or on the internet says that there is a 60% chance of rain, this probability has been determined by running hundreds of simulations through an iterative process.



24.1 Algebraic fractions

This section will show you how to:

- simplify algebraic fractions
- solve equations containing algebraic fractions.

Key word

algebraic fraction

Algebraic fractions can be added, subtracted, multiplied or divided using the same rules that apply to numbers.

To add and subtract, find a common denominator and then find equivalent fractions with that denominator.

Addition: For $\frac{a}{b} + \frac{c}{d}$, the common denominator is bd, so $\frac{a}{b} + \frac{c}{d} = \frac{ad}{bd} + \frac{bc}{bd}$

$$= \frac{ad + bc}{bd}$$

Subtraction: For $\frac{a}{b} - \frac{c}{d}$, the common denominator is bd, so $\frac{a}{b} - \frac{c}{d} = \frac{ad}{bd} - \frac{bc}{bd}$

$$= \frac{ad - bc}{bd}$$

This method works for more than two terms.

For example, for $\frac{a}{b} + \frac{c}{d} - \frac{e}{f}$, the common denominator is bdf, so

$$\frac{a}{b} + \frac{c}{d} - \frac{e}{f} = \frac{adf}{bdf} + \frac{bcf}{bdf} - \frac{bde}{bdf}$$

$$= \frac{adf + bcf - bde}{bdf}$$

To multiply, cancel any common factors, then multiply the numerators together and the denominators together.

Multiplication: $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$

To divide, find the reciprocal of the fraction you are dividing by, and then multiply.

Division: $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$

$$= \frac{ad}{bc}$$

Note that a, b, c and d can be numbers, other letters or algebraic expressions. Remember to:

- use brackets, if necessary, to avoid problems with signs and help you expand expressions
- factorise if you can
- cancel if you can.

Example 1

Simplify these fractions. **a** $\frac{1}{x} + \frac{x}{2y}$ **b** $\frac{2}{b} - \frac{a}{2b}$

a Common denominator is $2xy$:

$$\frac{1}{x} + \frac{x}{2y} = \frac{(1)(2y) + (x)(x)}{(x)(2y)}$$

$$= \frac{2y + x^2}{2xy}$$

b Common denominator is $2b$:

$$\frac{2}{b} - \frac{a}{2b} = \frac{4}{2b} - \frac{a}{2b}$$

$$= \frac{4 - a}{2b}$$

Example 2

Simplify these fractions. **a** $\frac{x}{3} \times \frac{x+2}{x-2}$ **b** $\frac{x}{3} \div \frac{2x}{7}$

a Multiplying: $\frac{x}{3} \times \frac{x+2}{x-2} = \frac{(x)(x+2)}{(3)(x-2)}$

$= \frac{x^2+2x}{3x-6}$

Remember, the line that separates the top and bottom of an algebraic fraction acts as brackets as well as a division sign.

b Dividing: $\frac{x}{3} \div \frac{2x}{7} = \frac{(x)(7)}{(3)(2x)}$

$= \frac{7}{6}$

Example 3

Solve this equation. $\frac{x+1}{3} - \frac{x-3}{2} = 1$

Subtract the fractions on the left-hand side: $\frac{(2)(x+1)-(3)(x-3)}{(2)(3)} = 1$

Multiply both sides by 6: $2(x+1) - 3(x-3) = 1(2)(3)$

Use brackets to avoid problems with signs and help you to expand to get a linear equation.

$2x + 2 - 3x + 9 = 6$

$-x = -5$

$x = 5$

Example 4

a Show that the equation $\frac{3}{x-1} - \frac{2}{x+1} = 1$ can be rewritten as $x^2 - x - 6 = 0$.

b Hence solve the equation $\frac{3}{x-1} - \frac{2}{x+1} = 1$.

a Add the fractions: $\frac{3(x+1)-2(x-1)}{(x-1)(x+1)} = 1$

Multiply both sides by the denominator: $3(x+1) - 2(x-1) = (x-1)(x+1)$

(Use brackets to help with expanding and to avoid problems with minus signs.)

Expand: $3x + 3 - 2x + 2 = x^2 - 1$

(Note that the right-hand side is the difference of two squares.)

Rearrange into the general quadratic form: $x^2 - x - 6 = 0$

b Factorise and solve: $(x-3)(x+2) = 0$

$x = 3$ or -2

Example 5

Simplify this fraction. $\dfrac{(x+6)(x+2)}{x+3} - \dfrac{(x+9)(x+1)}{x+5}$

Common denominator is $(x+3)(x+5)$: $\dfrac{(x+6)(x+2)(x+5)-(x+9)(x+1)(x+3)}{(x+3)(x+5)}$

Expand the brackets in the numerator and simplify: $\dfrac{(x^2+8x+12)(x+5)-(x^2+10x+9)(x+3)}{(x+3)(x+5)}$

$$= \frac{(x^3+8x^2+12x+5x^2+40x+60)-(x^3+10x^2+9x+3x^2+30x+27)}{(x+3)(x+5)}$$

$$= \frac{x^3+13x^2+52x+60-x^3-13x^2-39x-27}{(x+3)(x+5)}$$

$$= \frac{13x+33}{(x+3)(x+5)}$$

Note: It is sometimes simpler to leave an algebraic fraction in a factorised form.

Example 6

Simplify this expression. $\dfrac{x+3}{x^2+5x+4} - \dfrac{x-1}{x^2+4x}$

Factorise the denominators: $\dfrac{x+3}{(x+4)(x+1)} - \dfrac{x-1}{x(x+4)}$

Common denominator is $x(x+1)(x+4)$: $\dfrac{x(x+3)-(x-1)(x+1)}{x(x+1)(x+4)}$

Expand and simplify: $\dfrac{x^2+3x-(x^2-1)}{x(x+1)(x+4)}$

$$= \frac{x^2+3x-x^2+1}{x(x+1)(x+4)}$$

$$= \frac{3x+1}{x(x+1)(x+4)}$$

Example 7

Simplify this expression. $\dfrac{2x^2+x-3}{4x^2-9}$

Factorise the numerator and denominator: $\dfrac{(2x+3)(x-1)}{(2x+3)(2x-3)}$ (Denominator is the difference of two squares.)

Cancel common factors: $\dfrac{\cancel{(2x+3)}(x-1)}{\cancel{(2x+3)}(2x-3)}$

If at this stage there isn't a common factor on the top and bottom, you should check your factorisations.

The remaining fraction is the answer: $\dfrac{(x-1)}{(2x-3)}$

Example 8

Simplify this expression. $\dfrac{x+5}{\sqrt{x}-3} \times \dfrac{x^2-7x-18}{\sqrt{x}+3}$

Factorise the quadratic expression: $\dfrac{x+5}{\sqrt{x}-3} \times \dfrac{(x-9)(x+2)}{\sqrt{x}+3}$

Multiply: $\dfrac{(x+5)(x-9)(x+2)}{\left(\sqrt{x}-3\right)\left(\sqrt{x}+3\right)}$

$= \dfrac{(x+5)(x-9)(x+2)}{x-3\sqrt{x}+3\sqrt{x}-9}$

$= \dfrac{(x+5)(x-9)(x+2)}{x-9}$

$= (x+5)(x+2)$

$= x^2+7x+10$

Exercise 24A

1 Simplify each of these.

a $\dfrac{x}{2}+\dfrac{x}{3}$ **b** $\dfrac{3x}{4}+\dfrac{2x}{5}$ **c** $\dfrac{xy}{4}+\dfrac{2}{x}$

d $\dfrac{x+1}{2}+\dfrac{x+2}{3}$ **e** $\dfrac{x}{5}+\dfrac{2x+1}{3}$ **f** $\dfrac{x-4}{4}+\dfrac{2x-3}{2}$

2 Simplify each of these.

a $\dfrac{3x}{4}-\dfrac{x}{5}$ **b** $\dfrac{x}{2}-\dfrac{y}{3}$ **c** $\dfrac{xy}{4}-\dfrac{2}{y}$

d $\dfrac{2x+1}{2}-\dfrac{3x+1}{4}$ **e** $\dfrac{x-2}{2}-\dfrac{x-3}{4}$ **f** $\dfrac{x-4}{4}-\dfrac{2x-3}{2}$

3 Solve the following equations.

a $\dfrac{x+1}{2}+\dfrac{x+2}{5}=3$ **b** $\dfrac{4x+1}{3}-\dfrac{x+2}{4}=2$

c $\dfrac{2x+1}{2}-\dfrac{x+1}{7}=1$ **d** $\dfrac{3x+1}{5}-\dfrac{5x-1}{7}=0$

4 Simplify each of these.

a $\dfrac{x}{2}\times\dfrac{x}{3}$ **b** $\dfrac{4x}{3y}\times\dfrac{2y}{x}$ **c** $\dfrac{x}{2}\times\dfrac{x-2}{5}$

d $\dfrac{x}{5}\times\dfrac{2x+1}{3}$ **e** $\dfrac{x-5}{10}\times\dfrac{5}{x^2-5x}$

5 Simplify each of these.

a $\dfrac{2x}{7}\div\dfrac{4y}{14}$ **b** $\dfrac{4y^2}{9x}\div\dfrac{2y}{3x^2}$ **c** $\dfrac{x-3}{15}\div\dfrac{5}{2x-6}$

d $\dfrac{2x+1}{2}\div\dfrac{4x+2}{4}$ **e** $\dfrac{x}{6}\div\dfrac{2x^2+x}{3}$ **f** $\dfrac{x-2}{12}\div\dfrac{4}{x-3}$

g $\dfrac{x-5}{10}\div\dfrac{x^2-5x}{5}$

6 Simplify each of these. Factorise and cancel where appropriate.

a $\frac{3x}{4} + \frac{x}{4}$ **b** $\frac{3x}{4} - \frac{x}{4}$ **c** $\frac{3x}{4} \times \frac{x}{4}$

d $\frac{3x}{4} \div \frac{x}{4}$ **e** $\frac{3x+1}{2} + \frac{x-2}{5}$ **f** $\frac{3x+1}{2} - \frac{x-2}{5}$

g $\frac{3x+1}{2} \times \frac{x-2}{5}$ **h** $\frac{x^2-9}{10} \times \frac{5}{x-3}$ **i** $\frac{2x+3}{5} \div \frac{6x+9}{10}$

j $\frac{2x^2}{9} - \frac{2y^2}{3}$

CM **7** Show that each algebraic fraction simplifies to the given expression.

a $\frac{2}{x+1} + \frac{5}{x+2} = 3$ simplifies to $3x^2 + 2x - 3 = 0$

b $\frac{3}{4x+1} - \frac{4}{x+2} = 2$ simplifies to $8x^2 + 31x + 2 = 0$

c $\frac{2}{2x-1} - \frac{6}{x+1} = 11$ simplifies to $22x^2 + 21x - 19 = 0$

8 **a** Simplify this expression. $\frac{x+2\sqrt{2}}{x+\sqrt{2}} \times \frac{x-2\sqrt{2}}{x-\sqrt{2}}$

b Use your answer to find the value of $\frac{x+2\sqrt{2}}{x+\sqrt{2}} \times \frac{x-2\sqrt{2}}{x-\sqrt{2}}$ when $x = 1$.

9 Write $\frac{2}{x+3} + \frac{3}{x+4} - \frac{4}{x+5}$ as a single fraction with an expanded denominator.

10 Simplify this expression. $\frac{x^2+5x+6}{x+2} \div \frac{x^2-x-30}{x+5}$

EV **11** For homework a teacher asks her class to simplify the expression $\frac{x^2-x-2}{x^2+x-6}$.

This is Tom's answer:

$$\frac{\cancel{x^2} - x - \cancel{2}^{-1}}{\cancel{x^2} + x - \cancel{6}_{+3}} = \frac{-x-1}{x+3}$$

$$= \frac{x+1}{x+3}$$

Tom has made several mistakes. What are they?

MR **12** An expression of the form $\frac{ax^2+bx-c}{dx^2-9}$ simplifies to $\frac{x-1}{2x-3}$.

What was the original expression?

13 Solve the following equations.

a $\frac{4}{x+1} + \frac{5}{x+2} = 2$ **b** $\frac{18}{4x-1} - \frac{1}{x+1} = 1$ **c** $\frac{2x-1}{2} - \frac{6}{x+1} = 1$

d $\frac{3}{2x-1} - \frac{4}{3x-1} = 1$

14 Simplify the following expressions.

a $\frac{x^2+2x-3}{2x^2+7x+3}$ **b** $\frac{4x^2-1}{2x^2+5x-3}$ **c** $\frac{6x^2+x-2}{9x^2-4}$

d $\frac{4x^2+x-3}{4x^2-7x+3}$ **e** $\frac{4x^2-25}{8x^2-22x+5}$

CM **15** **a** Prove that the equation $\frac{7}{x+1} + \frac{4}{x+4} = 3$ simplifies to $3x^2 + 4x - 20 = 0$.

b Hence solve the equation $\frac{7}{x+1} + \frac{4}{x+4} = 3$.

EV **16** Emma swam 200 metres at a speed of $(x - 3)$ m/s, then a further 300 metres at $(x - 2)$ m/s.

The total swim took Emma 200 seconds.

a Show that $2x^2 - 15x + 25 = 0$.

b Find the speed at which Emma swam the first 200 metres.

Hints and tips Remember how speed, distance and time are related.

CM **17** **a** Expand and simplify $\left(x + \sqrt{2}\right)^3$.

b Hence show that $\left(1 + \sqrt{2}\right)^3 = 7 + 5\sqrt{2}$.

c Find the value of $\left(1 + \sqrt{2}\right)^6$.

18 Simplify these expressions.

a $\frac{x+4}{(x+3)^2} - \frac{x-4}{(x-3)^2}$ **b** $\frac{4}{x+1} + \frac{3}{x+2} - \frac{2}{x+3} - \frac{1}{x+4}$

PS **19** Simplify this expression. $\frac{x^2 + x\sqrt{3} - 6}{x^2 - x\sqrt{3} - 18}$

24.2 Changing the subject of a formula

This section will show you how to:

- change the subject of a formula where the subject occurs more than once.

When studying algebraic manipulation, you considered how to change the subject of a formula where the subject only appears once. To rearrange formulae where the subject appears more than once, the principle is the same as rearranging a formula where the subject only appears once or solving an equation where the unknown appears on both sides.

Collect all the subject terms on the same side and everything else on the other side. Most often, you then need to factorise the subject out of the resulting expression.

Example 9

Make x the subject of this formula. $ax + b = cx + d$

First, rearrange the formula to get all the x-terms on the left-hand side and all the other terms on the right-hand side: (The rule 'change sides – change signs' still applies.)

$ax - cx = d - b$

Factorise x out of the left-hand side: $x(a - c) = d - b$

Divide by the expression in brackets $x = \frac{d - b}{a - c}$

Example 10

Make p the subject of this formula. $5 = \frac{ap + b}{cp + d}$

First, multiply both sides by the denominator of the algebraic fraction: $5(cp + d) = ap + b$

Expand the brackets: $5cp + 5d = ap + b$

Now continue as in Example 9: $5cp - ap = b - 5d$

$p(5c - a) = b - 5d$

$p = \frac{b - 5d}{5c - a}$

Exercise 24B

1 Make c the subject of each formula.

a $5(c - 3) = p$ **b** $5(c - 3) = cp$

2 Make G the subject of each formula.

a $F = \frac{R}{G + 3}$ **b** $F = \frac{G + R}{G + 3}$

3 Make the letter in brackets the subject of the formula.

a $p(a + b) = q(a - b)$ (a) **b** $p(a + b) = q(a - b)$ (b)

c $5 = \frac{a + b}{a - c}$ (a) **d** $A = 2\pi rh + \pi rk$ (r)

e $v^2 = u^2 + av^2$ (v) **f** $R = \frac{x - 3}{x - 2}$ (x)

4 **a** The perimeter of a shape is given by the formula $P = \pi r + 2kr$. Make r the subject of this formula.

b The area of the same shape is given by $A = \frac{1}{2}\,[\pi r^2 + r^2\sqrt{(k^2 - 1)}\,]$. Make r the subject of this formula.

5 When £P is invested for Y years at a simple interest rate of R, the following formula gives the amount, A, at any time.

$$A = P + \frac{PRY}{100}$$

Make P the subject of this formula.

6 When two resistors with values a and b are connected in parallel, the total resistance is given by:

$$R = \frac{ab}{a + b}$$

a Make b the subject of the formula.

b Write the formula when a is the subject.

EV **7** **a** Make x the subject of this formula. $y = \frac{x + 2}{x - 2}$

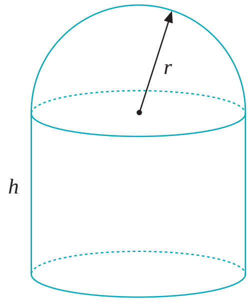
b Show that the formula $y = 1 + \frac{4}{x - 2}$ can be rearranged to give $x = 2 + \frac{4}{y - 1}$.

c Combine the right-hand sides of each formula in part **b** into single fractions and simplify as much as possible.

d What do you notice?

EV **8** The volume of the solid shown is given by this formulae.

$V = \frac{2}{3}\pi r^3 + \pi r^2 h$



a Explain why it is not possible to make r the subject of this formula.

b Make π the subject.

c If $h = r$, can the formula be rearranged to make r the subject? If so, rearrange it to make r the subject.

9 Make x the subject of this formula. $W = \frac{1}{2}z(x + y) + \frac{1}{2}y(x + z)$

EV **10** The following formulae in x can be rearranged to give the formulae in terms of y as shown.

$y = \dfrac{x + 1}{x + 2}$ gives $x = \dfrac{1 - 2y}{y - 1}$ $\qquad y = \dfrac{2x + 1}{x + 2}$ gives $x = \dfrac{1 - 2y}{y - 2}$

$y = \dfrac{3x + 2}{4x + 1}$ gives $x = \dfrac{2 - y}{4y - 3}$ $\qquad y = \dfrac{x + 5}{3x + 2}$ gives $x = \dfrac{5 - 2y}{3y - 1}$

Without rearranging the formula, write down $y = \dfrac{5x + 1}{2x + 3}$ as $x = \ldots$ and explain how you can do this without any algebra.

EV **11** Alice and Brian have been asked to make u the subject of the formula $\dfrac{1}{f} = \dfrac{1}{u} + \dfrac{1}{v}$.

Alice's answer is $u = \dfrac{fv}{v - f}$.

Brian's answer is $u = \dfrac{1}{\frac{1}{f} - \frac{1}{v}}$.

a Evaluate whether either or both of these answers are correct.

b Into which answer is it easier to substitute values?

24.3 Functions

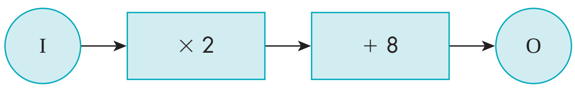
This section will show you how to:

- find the output of a function
- find the inverse function.

Key word

inverse function

This is a function machine.



When you input a number into the function machine, it doubles the number and then adds 8 to produce an output. For example, if the input is 7, then the output is $7 \times 2 + 8 = 14 + 8 = 22$.

If you are told the output, then you can also find the input by applying the inverse operations in reverse order. For example, if the output is 30, then you subtract 8 and divide by 2.

$30 - 8 = 22 \qquad 22 \div 2 = 11$

So the input was 11.

Example 11

Use this function machine to find the output for each the input.

a 6 **b** – 4

c Find both possible inputs if the output is 20.

$$\text{I} \rightarrow \boxed{\text{SQUARE}} \rightarrow \boxed{-\ 5} \rightarrow \text{O}$$

a $6^2 = 36$ $36 - 5 = 31$

b $(-4)^2 = 16$ $16 - 5 = 11$

c $20 + 5 = 25$ $\sqrt{25} = 5$ or -5

Another way of writing a function is to use the function notation $f(x)$. The value of x that you substitute into a function $f(x)$ is the input and the value of $f(x)$ is the output of the function. For example, for the function $f(x) = 3x + 10$, when the input is 5, the output $f(5)$ is $3 \times 5 + 10 = 35$.

Example 12

The function $f(x)$ is defined as $f(x) = 14 - 2x$.

Find: **a** $f(3)$ **b** $f(-5)$. **c** Solve $f(x) = 0$.

a $f(3) = 14 - 2 \times 3$
$= 14 - 6$
$= 8$

b $f(-5) = 14 - 2 \times (-5)$
$= 14 + 10$
$= 24$

c When $f(x) = 0$, $14 - 2x = 0$,
so $x = 7$.

Exercise 24C

1 **a** Use this function machine to find the output for each input.

i 11 **ii** 26 **iii** –4 **iv** 1

$$\text{I} \rightarrow \boxed{+\ 9} \rightarrow \boxed{\div\ 2.5} \rightarrow \text{O}$$

b Find the input for each of these outputs.

i 18 **ii** 0 **iii** 500 **iv** 3

2 **a** Use the function machine to find the output when the input is 29.

$$\text{I} \rightarrow \boxed{+\ 7} \rightarrow \boxed{\div\ 4} \rightarrow \boxed{-\ 3} \rightarrow \text{O}$$

b Find an algebraic expression for the output when the input is x.

c Use your answer for **b** to find the input when the output is 10.

3 For this function machine, find the input for each of these outputs.

a 30 **b** 60 **c** 18

$$\text{I} \rightarrow \boxed{-\ 2} \rightarrow \boxed{\text{cube root}} \rightarrow \boxed{\times\ 10} \rightarrow \text{O}$$

MR **4** For which input do these function machines also have the same output?

$$\text{I} \rightarrow \boxed{+\,3} \rightarrow \boxed{\times\,4} \rightarrow \text{O}$$

$$\text{I} \rightarrow \boxed{\times\,6} \rightarrow \boxed{-\,2} \rightarrow \text{O}$$

5 $f(x) = 2x^2 + 7$ Find the value of each of these for $f(x)$.

a $f(3)$ **b** $f(11)$ **c** $f(-2)$

d $f(30)$ **e** $f(-30)$ **f** $f(\sqrt{3})$

g Given that $f(k) = 57$, find both values of k.

6 $g(x) = 25 - x^2$ Find the value of each of these for $g(x)$.

a $g(4)$ **b** $g(-8)$ **c** $g(9)$

d $g(-9)$ **e** $g(\sqrt{13})$ **f** $g(-0.4)$

g Solve $g(x) = 0$.

MR **7** **a** Given that $f(x) = 2x^2 + 3x$, find the value of:

i $f(-6)$ and **ii** $f(4)$.

$g(x) = x^2 + 8x + 6$

b Find the inputs for which $f(x)$ and $g(x)$ have the same outputs.

Inverse functions

An **inverse function** is a function that performs the opposite process of the original function, such as adding instead of subtracting or multiplying instead of dividing. If the original function turns an input of 3 into an output of 5, then the inverse function turns the output of 5 back into the input of 3, and it will do this for all inputs and outputs. The notation used for an inverse function is $f^{-1}(x)$.

For example, the inverse function of $f(x) = 2x + 1$ is $f^{-1}(x) = \dfrac{x-1}{2}$.

To find the inverse function, write $f(x)$ as y; make x the subject of the function; replace x with $f^{-1}(x)$ and then replace y with x.

Consider $f(x) = 2x + 1$.

Writing (x) as y gives $y = 2x + 1$.

Making x the subject of the function gives $x = \dfrac{y-1}{2}$.

Replacing x with $f^{-1}(x)$ and then y with x gives $f^{-1}(x) = \dfrac{x-1}{2}$.

Example 13

Find the inverse of the function $f(x) = x^3 + 10$.

Write $f(x)$ as y:	$y = x^3 + 10$
Subtract 10 from both sides:	$y - 10 = x^3$
Cube root each side:	$\sqrt[3]{y-10} = x$
Reverse the sides:	$x = \sqrt[3]{y-10}$
Replace x with $f^{-1}(x)$ and then y with x:	$f^{-1}(x) = \sqrt[3]{x-10}$

Exercise 24D

1 Find an expression for $f^{-1}(x)$ for:

a $f(x) = 4x - 5$ **b** $f(x) = x^3 + 2$ **c** $f(x) = \dfrac{10}{x + 1}$

d $f(x) = 10 - 2x$ **e** $f(x) = \dfrac{x - 7}{6}$ **f** $f(x) = \dfrac{3}{x} + 5$

2 **a** Given that $f(x) = \dfrac{x + 2}{3x - 5}$, find an expression for $f^{-1}(x)$.

b Find the value of f(1).

c Substitute f(1) into $f^{-1}(x)$ to verify that the answer is 1.

MR **3** **a** Find the inverse functions of

i $f(x) = 12 - x$ and **ii** $g(x) = \dfrac{12}{x}$.

What do you notice?

b Find the inverse function of $f(x) = \dfrac{3x + 8}{4x - 3}$. What do you notice?

c Prove that if $f(x) = \dfrac{ax + b}{cx - a}$, then $f^{-1}(x) = \dfrac{ax + b}{cx - a}$.

24.4 Composite functions

This section will show you how to:

- find the composite of two functions.

Key word

composite

A **composite** function is a combination of two functions to create a third function. For two functions $f(x)$ and $g(x)$, the function created by substituting $g(x)$ into $f(x)$ is called $fg(x)$. You work this out by finding $g(x)$ first and then substituting your answer into $f(x)$.

Example 14

The functions $f(x)$ and $g(x)$ are defined as $f(x) = 5x - 3$ and $g(x) = \frac{1}{2}x + 1$. Find each of the following.

a f(4) **b** fg(4) **c** ff(4)

a $f(4) = 5 \times 4 - 3$
$= 20 - 3$
$= 17$

b $g(4) = \frac{1}{2} \times 4 + 1$
$= 2 + 1$
$= 3$

$f(3) = 5 \times 3 - 3$
$= 15 - 3$
$= 12$

c $f(4) = 17$

$f(17) = 5 \times 17 - 3$
$= 85 - 3$
$= 82$

Example 15

The functions $f(x)$ and $g(x)$ are defined as $f(x) = 5x - 3$ and $g(x) = \frac{1}{2}x + 1$. Find each of the following.

a $fg(x)$ **b** $gf(x)$ **c** $ff(x)$

a Substitute $g(x)$ into $f(x)$:
$f(x) = 5(\frac{1}{2}x + 1) - 3$
$= 2\frac{1}{2}x + 5 - 3$
$= 2\frac{1}{2}x + 2$

b Substitute $f(x)$ into $g(x)$:
$g(x) = \frac{1}{2}(5x - 3) + 1$
$= 2\frac{1}{2}x - 1\frac{1}{2} + 1$
$= 2\frac{1}{2}x - \frac{1}{2}$

c Substitute $f(x)$ into $f(x)$:
$f(x) = 5(5x - 3) - 3$
$= 25x - 15 - 3$
$= 25x - 18$

Exercise 24E

1 Given that $f(x) = 10 - 3x$ and $g(x) = \frac{x-3}{2}$, find the value of each of the following.

a fg(4) **b** gf(–2) **c** ff(7) **d** gg(0) **e** fgfgfg(5.8)

Hints and tips For fg(a), calculate g(a) and substitute this into f(x).

2 Given that $f(x) = \frac{24}{x-1}$ and $g(x) = 5x - 11$, find the value of each of the following.

a fg(2.5) **b** gf(1.5) **c** fgf(10) **d** gfg(3) **e** ff(–11) **f** gg(–3)

CM **3** $f(x) = x^3 - 6g(x) = 4(x - 2)$ $h(x) = 3 - x$

a Find simplified expressions for each of the following.

i gf(x) **ii** hg(x) **iii** fh(x) **iv** gg(x) **v** ff(x)

b Prove that gh(x) can never equal hg(x) for any value of x.

Hints and tips For gf(x), substitute f(x) into g(x).

PS **4** $f(x) = x^2 + a$ $g(x) = x - b$

If fg(x) = gf(x), find x in terms of b.

24.5 Iteration

This section will show you how to:

- find an approximate solution for an equation using the process of iteration.

Key word

iteration

Many equations cannot be solved exactly using any of the techniques you have met already. You could use trial and improvement to solve an equation like this but there is a process called **iteration** which is more efficient and does not require you to make a new decision after each attempt. This involves solving the equation many times, using your result from the previous version each time to make the answer more accurate.

To perform iteration, first rearrange the equation so that x is the subject, although there will be x terms on the other side (the right hand-side) as well. The x that is the subject is called x_{n+1} and any x term on the right hand side is called x_n.

For example, $x_{n+1} = \sqrt{2x_n + 6}$ can be used to solve the quadratic equation $x^2 - 2x - 6 = 0$.

Substitute an initial value, called x_1, into the right-hand side, and call the value obtained from this substitution x_2.

Example 16

Find the first four iterations of the iterative formula $x_{n+1} = 3x_n - 2$ with $x_1 = 2$.

$x_2 = 3x_1 - 2$	$x_3 = 3x_2 - 2$	$x_4 = 3x_3 - 2$	$x_5 = 3x_4 - 2$
$= 3 \times 2 - 2$	$= 3 \times 4 - 2$	$= 3 \times 0 - 2$	$= 3 \times 28 - 2$
$= 4$	$= 10$	$= 28$	$= 82$

Using a calculator makes iteration much easier.

To solve the quadratic equation $x^2 - 2x - 6 = 0$ using $x_{n+1} = \sqrt{2x_n + 6}$:

Let $x_1 = 4$.

On your calculator, type 4 =. This records the number 4 as the first 'answer'.

Next type √ 2 (× Ans + 6) =

Note: your calculator may work in a slightly different way, and require a different key press order. Check how this works on your calculator before starting any iteration questions.

$x_2 = \sqrt{2x_1 + 6} = \sqrt{14} = 3.7417$ (4 dp)

You then substitute this value back into the right hand side to generate the term x_3, and so on.

$x_3 = \sqrt{2x_2 + 6} = \sqrt{2 \times 3.7417 + 6} = 3.6720$ (4 dp)

$x_4 = \sqrt{2x_3 + 6} = \sqrt{2 \times 3.6720 + 6} = 3.6529$ (4 dp)

You can just keep pressing = on your calculator to generate further iterations.

Example 17

An approximate solution for the equation $x^3 - 16x + 9 = 0$ can be found using the iterative formula $x_{n+1} = \sqrt[3]{16x_n - 9}$ and an initial value of $x_1 = 4$.

a Find the first six iterations, correct to 5 decimal places.

b Verify that 3.68 is a solution of the equation, correct to 2 decimal places.

a Enter the initial value on the calculator: 4 =

Enter the iterative formula: ∛ (1 6 × Ans − 9) =

This substitutes $x_1 = 4$ into the formula: $x_2 = 3.80295$

Press = to substitute the value of x_2 into the formula: $x_3 = 3.72885$

Pressing = four more times gives: $x_4 = 3.70021$ $x_5 = 3.68902$ $x_6 = 3.68463$ $x_7 = 3.68290$

b Both x_6 and x_7 round to 3.68, correct to 2 decimal places.

Example 18

These steps can be used to find an approximate value for $x^3 = 6x + 8$.

Step 1: Start with $x = 3$.

Step 2: Find the value of $\sqrt[3]{6x + 8}$, correct to 4 decimal places.

Step 3: Compare your answer with the value of x you substituted. If it is the same, you have found the answer. If it is not the same, go back to step 2.

Find the solution of $x^3 = 6x + 8$ given by this process.

Enter the initial value on the calculator: 3 =

Enter the iterative formula: ∛ (6 × Ans + 8) =

First iteration = $\sqrt[3]{6 \times 3 + 8} = 2.9625$ (4 dp)	Not the same as 3 so return to step 2
Second iteration = 2.9539 (4 dp)	Not the same as 2.9625 so return to step 2
Third iteration = 2.9520 (4 dp)	Not the same as 2.9539 so return to step 2
Fourth iteration = 2.9515 (4 dp)	Not the same as 2.9520 so return to step 2
Fifth iteration = 2.9514 (4 dp)	Not the same as 2.9515 so return to step 2
Sixth iteration = 2.9514 (4 dp)	The same as 2.9514 from the fifth iteration so this is the answer.

Example 19

For the iterative formula $x_{n+1} = \dfrac{3}{3 - x_n}$, find the value of x_{200} when $x_1 = 5$.

$x_2 = \dfrac{3}{3-5} = -\dfrac{3}{2}$ $\qquad x_3 = \dfrac{3}{3-\left(-\frac{3}{2}\right)} = \dfrac{3}{\frac{9}{2}} = \dfrac{2}{3}$

$x_4 = \dfrac{3}{3-\left(\frac{2}{3}\right)} = \dfrac{3}{\frac{7}{3}} = \dfrac{9}{7}$ $\qquad x_5 = \dfrac{3}{3-\left(\frac{9}{7}\right)} = \dfrac{3}{\frac{12}{7}} = \dfrac{7}{4}$

$x_6 = \dfrac{3}{3-\left(\frac{7}{4}\right)} = \dfrac{3}{\frac{5}{4}} = \dfrac{12}{5}$ $\qquad x_7 = \dfrac{3}{3-\left(\frac{12}{5}\right)} = \dfrac{3}{\frac{3}{5}} = 5$

Since you started with 5, this sequence will now cycle round 5, $-\frac{3}{2}$, $\frac{2}{3}$, $\frac{9}{7}$, $\frac{7}{4}$, $\frac{12}{5}$, returning to $\frac{12}{5}$ for every multiple of 6 (x_6, x_{12}, x_{18}, etc.).

The largest multiple of 6 below 200 is 198, which is 2 less than 200, so $x_{200}(5)$ will be the same as x_2.

Hence $x_{200} = -\dfrac{3}{2}$.

Exercise 24F

1 Find the first four iterations using the iterative formula $x_{n+1} = 18 - 7x_n$ with $x_1 = 2$.

2 For the iterative formula $x_{n+1} = 5x_n - 12$, find x_5 for each initial value.

a $x_1 = 6$ **b** $x_1 = -4$ **c** $x_1 = 1.2$ **d** $x_1 = 3$

3 These steps can be used to find an approximate value for $x^3 = 19x + 34$.

Step 1: Start with $x = 5$.

Step 2: Find the value of $\sqrt[3]{19x + 34}$, correct to 4 decimal places.

Step 3: Compare your answer with the value of x you substituted. If it is the same, you have found the answer. If it is not the same, go back to step 2.

Find the solution to $x^3 = 19x + 34$ given by this process.

Hints and tips Type in the value of x_1, then use the **Ans** button to type in the iterative formula.

4 An approximate solution for the equation $x^3 - 7x - 10 = 0$ can be found using the iterative formula $x_{n+1} = \sqrt[3]{7x_n + 10}$ and an initial value of $x_1 = 3$.

Find the first five iterations, correct to 4 decimal places.

CM **5** The iterative formula $x_{n+1} = \sqrt[3]{x_n^{\,2} + 5}$ can be used to find an approximate solution for the equation $x^3 - x^2 - 5 = 0$.

a Using $x_1 = 2$, show that $x_5 = 2.12$, correct to 2 decimal places.

b Verify that 2.12 is a solution, correct to 2 decimal places.

CM **6** Use the iterative formula $x_{n+1} = \sqrt[3]{5x_n + 2}$ to show that 2.4142 is an approximate solution for the equation $x^3 - 5x - 2 = 0$.

EV **7** **a** Solve, by factorisation, the equation $x^2 - 10x + 21 = 0$.

b Use the iterative formula $x_{n+1} = \sqrt{10x_n - 21}$ with $x_1 = 5$ to determine which of the two answers is generated by the formula.

c Investigate what happens with each of these initial values.

i $x_1 = 3.001$ **ii** $x_1 = 2.999$ **iii** $x_1 = 100$ **iv** $x_1 = 3$

d Generalise what happens for all values of x.

CM **8** The iterative formula $x_{n+1} = \sqrt[3]{x_n^{\,2} + 5}$ can be used to find an approximate solution for the equation $x^3 - x^2 - 5 = 0$.

a Show that the equation $x = \dfrac{x+4}{x-1}$ can be rearranged as $x^2 - 2x - 4 = 0$ and hence $x_{n+1} = \dfrac{x_n + 4}{x_n - 1}$ is an iterative formula for the equation $x^2 - 2x - 4 = 0$.

b When $x_1 = 3$, find the values of x_2, x_3 and x_4, writing the answers as fractions, where appropriate.

c When $x_1 = 3.25$, find the values of x_2, x_3 and x_4,writing the answers as fractions.

d When $x_1 = 3.3$, find the values of x_2, x_3 and x_4,writing the answers as fractions.

e When $x_1 = 1 + \sqrt{5}$, find the values of x_2, x_3 and x_4, writing the answers as surds.

f Use the results from parts **b** to **e** to deduce a solution for the equation $x^2 - 2x - 4 = 0$.

EV **9** A square has sides of $(x^2 - 2)$ cm and $(x + 5)$ cm.

a Show that $x^2 - x - 7 = 0$.

b Use the iterative formula $x_{n+1} = \sqrt{x_n + 7}$ and an initial input of $x_1 = 3$ to find the area of the square, correct to the nearest integer.

c How reliable is your answer for the area of the square?

PS **10** **a** For the iterative formula $x_{n+1} = x_n^{\,2} - 3$, find the value of x_{36} when $x_1 = 2$.

b For the iterative formula $x_{n+1} = \dfrac{1}{1 - x_n}$, find the value of x_{100} when $x_1 = 3$.

Hints and tips Generate x_2, x_3, x_4 and so on, and look for a pattern.

EV **11** The equation $x^3 = 4x + 9$ is to be solved by an iterative formula with $x_1 = 3$.

a Investigate what happens when the iterative formula used is given by

$$x_{n+1} = \frac{9}{x_n^{\,2} - 4}.$$

b Investigate what happens when the iterative formula used is given by

$$x_{n+1} = \frac{x_n^{\,3} - 9}{4}$$

c Investigate what happens when the iterative formula used is given by

$$x_{n+1} = \sqrt{4 + \frac{9}{x_n}}.$$

Worked exemplars

CM **1** The equation $x^3 - 19x + 9 = 0$ can be written as the iterative formula $x_{n+1} = \sqrt[3]{19x_n - 9}$.

a Using $x_1 = 4$, find the first two iterations, correct to 3 decimal places.

b Show that 4.10 is a solution to the equation, correct to 2 decimal places.

This is a question on communicating mathematics so you need to state your method clearly.	
a First iteration $x_2 = \sqrt[3]{19x_1 - 9} = \sqrt[3]{19 \times 4 - 9} = \sqrt[3]{67} =$ 4.062 (3 dp)	Start by finding the first iteration. Although the method shown is correct, it is more efficient to register 4 as an answer in the calculator's memory and then to type in the iterative formula using Ans for x_n, especially if you need more than two iterations.
Second iteration $x_3 = \sqrt[3]{19x_2 - 9} = \sqrt[3]{19 \times \sqrt[3]{67} - 9} =$ 4.085 (3 dp)	Then move on to the second iteration. Note that you should be using the exact answer from the first iteration rather than the rounded 4.062. (This would definitely happen if you were using the efficient calculator method.) In this example, 4.062 also gives a second iteration of 4.085, but this is not guaranteed.
b $x_4 = 4.094$ (3 dp) $x_5 = 4.097$ (3 dp) $x_6 = 4.099$ (3 dp) $x_7 = 4.099$ (3 dp)	Since you are asked to show that 4.1 is a solution, you are being asked to communicate information accurately. Continue to find iterations until two of them repeat.
Since x_6 and x_7 both equal 4.099 correct to 3 decimal places, 4.10 is a solution of the equation correct to 2 decimal places.	Include a conclusion.

PS **2** Given that $f(x) = \dfrac{x^2 + 3x - 10}{2x^2 - 9x + 10}$, prove that $f^{-1}(3) = 4$.

This is a problem solving question so you need to plan a strategy to solve it and, most importantly, communicate your method clearly. You need to show each step clearly.

There are two different methods shown here.

Method 1 $f(x) = \dfrac{x^2 + 3x - 10}{2x^2 - 9x + 10} = \dfrac{(x+5)(x-2)}{(2x-5)(x-2)}$ $f(x) = \dfrac{x+5}{2x-5}$	Factorise and cancel the numerator and denominator.
$y = \dfrac{x+5}{2x-5}$ $y(2x - 5) = x + 5$ $2xy - 5y = x + 5$ $2xy - x = 5y + 5$ $x(2y - 1) = 5y + 5$ $x = \dfrac{5y+5}{2y-1}$ $f^{-1}(x) = \dfrac{5x+5}{2x-1}$	Find the inverse function.
$f^{-1}(3) = \dfrac{5 \times 3 + 5}{2 \times 3 - 1} = \dfrac{20}{5} = 4$	Substitute $x = 3$.
Method 2 $3 = \dfrac{x^2 + 3x - 10}{2x^2 - 9x + 10}$	Put the function equal to 3.
$3(2x^2 - 9x + 10) = x^2 + 3x - 10$	Multiply by the denominator.
$6x^2 - 27x + 30 = x^2 + 3x - 10$ $5x^2 - 30x + 40 = 0$ $x^2 - 6x + 8 = 0$	Simplify.
$(x - 2)(x - 4) = 0$ $x = 2$ or 4	Factorise.
Not $x = 2$ because $f(x)$ is undefined for $x = 2$ in its original form (since both the numerator and denominator would equal zero, and division by zero is forbidden. $x = 4$, so $f^{-1}(3) = 4$	Explain why $x = 2$ would not be allowed.

Ready to progress?

I can find the output of a function given an input.

I can rearrange more complicated formulae where the subject may appear twice or as a power.

I can find an inverse function by rearranging.
I can find a composite function by combining two functions together.
I can combine and simplify algebraic fractions.
I can use iteration to find a solution to an equation to an appropriate degree of accuracy.

Review questions

1 $f(x) = 20 - 3x^2$. Find the value of $f(-2)$.

2 **a** Make x the subject of the formula $6x - K = a - Cx$.

b Hence find the value of x when $a = 5$, $K = -12$ and $C = -8$.

3 **a** Write $f(x) = \frac{x}{x-3} - \frac{9}{x(x-3)}$ as a single fraction in its simplest form.

b Hence find the inverse function $f^{-1}(x)$.

4 Simplify fully $\frac{21x^2 - 7x}{9x^2 - 1}$.

EV **5** The iterative formula $x_{n+1} = \sqrt[5]{6x_n^{\ 3} + 13}$ can be used to solve the equation $x^5 = 6x^3 + 13$.

a Starting with $x_1 = 2.5$, find the first four iterations, all correct to 2 decimal places.

b Find x_5 correct to 2 decimal places and compare it with x_4.

CM **6** $f(x) = 3x + 8$ $\qquad$ $g(x) = x^3 + 2$

a Find a simplified expression for $fg(x)$.

b Using the expression from part **a**, verify that $fg(3) = 95$.

7 Find the inverse of each function.

a $f(x) = px - q$ $\qquad$ **b** $f(x) = a - x^3$ $\qquad$ **c** $f(x) = \frac{a}{x + c}$

MR **8** $f(x) = \left(\sqrt{2} + \sqrt{x}\right)^2$.

a Find each of the following.

i $f(0)$ $\qquad$ **ii** $ff(0)$ $\qquad$ **iii** $fff(0)$ $\qquad$ **iv** $ffff(0)$ $\qquad$ **v** $fffff(0)$

b Find the nth term of the sequence given by the answers to part **a**.

CM **9** Show, by iteration, that a solution of the equation $x^3 = 2x + 2$ is given by 1.77, correct to 2 decimal places.

10 **a** Simplify $f(x) = \frac{2x^2 + 3x - 14}{x^2 - 5x + 6}$.

$g(x) = \frac{12 - x^2}{x}$

b Solve $gf(x) = 1$.

EV **11** Alex was working out $f^{-1}(x)$ for the function $f(x) = \dfrac{4 - x}{2 - 3x}$.

Find the mistakes in Alex's solution and write the correct solution.

$$y = \frac{4 - x}{2 - 3x}$$
$$y(2 - 3x) = 4 - x$$
$$2 - 3xy = 4 - x$$
$$-3xy = 2 - x$$
$$x - 3xy = 2$$
$$x(1 - 3y) = 2$$
$$x = \frac{1 - 3y}{2}$$

Hence $f^{-1}(x) = \dfrac{1 - 3x}{2}$

PS **12** $x_{n+1} = \dfrac{2}{2 - x_n}$

If $x_1 = 5$, find each of the following.

i x_{219} **ii** x_{238} **iii** x_{257} **iv** x_{276}

PS **13** Simplify fully $\dfrac{(x + 2)(x + 4)(x - 6) - x^3}{7x^2 + 19x + 12}$

PS **14** $f(x) = x^2 - 81$ $g(x) = 19 - x$

a Solve $f(x) = g(x)$, giving both answers correct to 3 significant figures.

b Solve $fg(x) \geqslant 0$.

c Solve $gf(x) > 0$.

PS **15** A right-angled triangle has a base of $(x + 8)$ cm and a perpendicular height of $(x - 2)$ cm. The area of the triangle is 22 cm^2.

a Show that $x^2 + 6x - 60 = 0$.

The equation $x^2 + 6x - 60 = 0$ can be rewritten as the iterative formula $x_{n+1} = \sqrt{60 - 6x_n}$.

b Find the value of x, correct to 3 significant figures.

CM **16** $f(x) = x^2 - 4x$ $g(x) = 2x + 3$

a Find fg(2).

b Show that if $fg(x) = gf(x)$, then the answer can be written in the form $a \pm b\sqrt{3}$.

PS **17** Margaret has n beads in a bag, of which 5 are green. She removes two beads at random from the bag at the same time.

The probability that neither bead is green is $\dfrac{7}{22}$.

a Show that $3n^2 - 47n + 132 = 0$.

b How many beads were in the bag originally?