2 OSCILLATION

PRIOR KNOWLEDGE

You will be familiar with Hooke's law (*Chapter 12 of Year 1 Student Book*) and know what is meant by the spring constant. You will have experience of using equations for elastic potential energy, gravitational potential energy and kinetic energy, and understand the principle of conservation of energy (Chapter 11 of *Year 1 Student* Book). You may want to refresh your memory about stationary waves on strings and in pipes (see *section 5.6 of Chapter 5 of Year 1 Student Book*). You will need to know how to determine sine and cosine values for angles expressed in radians. You will have already studied circular motion (Chapter 1 of this book) as a type of periodic motion.

LEARNING OBJECTIVES

In this chapter you will extend your knowledge of periodic motion to include oscillations and resonance. You will learn that many natural systems oscillate with simple harmonic motion (SHM), and you will analyse such motion graphically. You will find out about forced vibration in mechanical systems, and the effect of damping. You will have the opportunity to learn how to analyse experimental data using logarithms.

(Specification 3.6.1.2 to 3.6.1.4)

Many people have heard of 'MRI' and know that it is a type of scan that a person may have in hospital to help in the diagnosis and treatment of an injury or illness (Figure 1). Some people will know that MRI stands for 'magnetic resonance imaging', but few will know that MRI is a medical application of an effect called 'nuclear magnetic resonance' (NMR). Resonance is all about efficiently transferring energy from one vibrating system to another. It occurs in many areas of physics and in everyday life, from lasers to musical instruments. In NMR, a nucleus, in the presence of a very strong magnetic field, is able to gain energy by absorbing radio-frequency electromagnetic radiation (typically 60 to 1000 MHz).

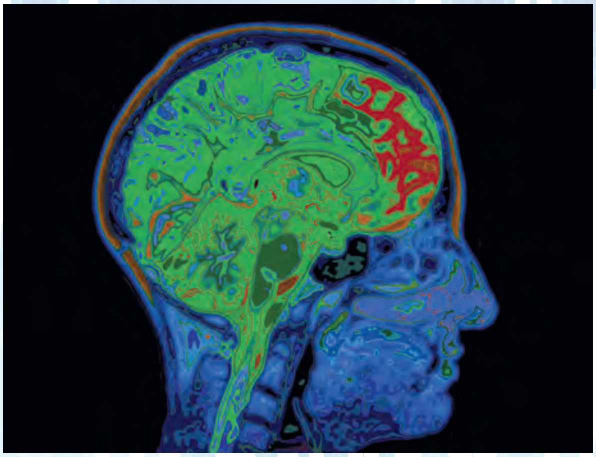

Figure 1 *MRI scan image of the human brain*

When a patient has an MRI scan, he or she is required to lie down inside a large tube-shaped chamber. A powerful magnetic field is activated in the chamber, enabling the transfer of energy from the radio-frequency radiation to hydrogen nuclei (protons) in specific body tissues. The controlled de-excitation of the protons enables the mapping of the tissue, in terms of the hydrogen atoms present.

2.1 SIMPLE HARMONIC MOTION

Oscillations and vibrations are a type of **periodic motion**. The motion repeats in a regular way as time passes. An object 'oscillates' when it repeatedly moves backwards and forwards about an equilibrium position. This **mechanical oscillation** (as opposed to the oscillation of fields, as in an electromagnetic wave) requires the action of a resultant force that is *always directed towards the equilibrium position*. The resultant force in this type of motion is often referred to as the **restoring force**, and the distance and direction of the oscillating object from its equilibrium position are its **displacement**, x.

Imagine holding a spring vertically, with a mass attached to its lower end. Initially, it is stationary – it is in equilibrium. If you pull the mass downwards, a restoring force acts upwards. When the mass moves up above its equilibrium position, the restoring force acts downwards. The result is an oscillatory motion in which the displacement x varies periodically.

A special kind of oscillation is **simple harmonic motion** (SHM):

In simple harmonic motion (SHM), the restoring force is directly proportional to the displacement, and in the opposite direction.

The repeated up-and-down motion of a mass on a spring is an example of SHM and can be monitored using data logging equipment connected to a motion sensor (Figure 2). Computer analysis of the data shows that the displacement versus time graph is a cosine (or sine) curve.

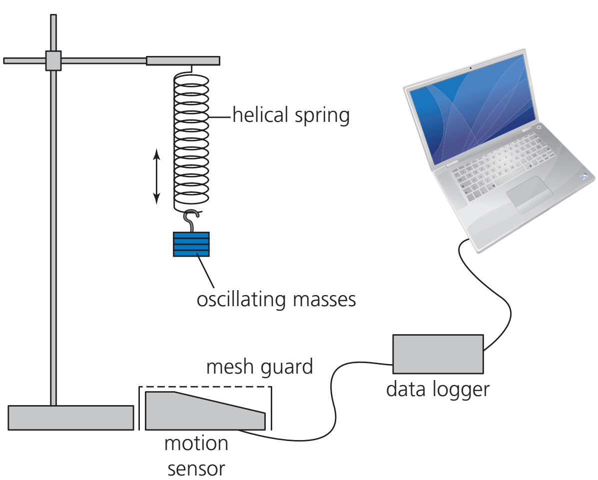

Figure 2 *A motion sensor with data logger and computer for analysis of the oscillatory motion of a mass on a spring*

The graph of displacement versus time for any SHM, in which the displacement is at its maximum value at $t = 0$, is shown in Figure 3. The equation of the graph is

$$x = A\cos(\omega t)$$

Here A is **amplitude** of the oscillation, which is the maximum displacement, and ω is a constant defined by $\omega = \frac{2\pi}{T}$, where T is the constant time period, which is the time for one **cycle** of oscillation. A cycle corresponds to the oscillating mass moving through any position then passing through that same position again in the same direction. For example, the mass at P in Figure 3 has displacement X. It passes through the equilibrium position, moves to one extreme, then to the other extreme, and then at Q has the same displacement X and moves in the same direction. Thus P to Q is one cycle. The number of cycles of oscillation per second is the frequency, f, measured in hertz ($\mathrm{Hz} = \mathrm{s}^{-1}$), which is related to the time period by $f = \frac{1}{T}$. Therefore ω can also be written as $\omega = 2\pi f$. It is called the **angular frequency** of the oscillation and is measured in $\mathrm{rad\,s^{-1}}$.

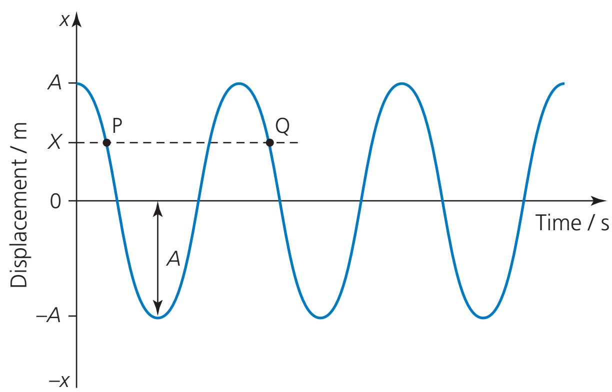

Figure 3 *Displacement versus time graph (cosine curve) for SHM with $x = A$ when $t = 0$*

The equation $x = A\cos(\omega t)$ representing the displacement of an object oscillating with SHM is correct provided the object is at an extreme position when the oscillation starts to be monitored – so, when $t = 0$, $x = A$. However, if the oscillating object is passing through its mean position when the oscillation starts to be monitored, then at $t = 0$, $x = 0$, and the graph of displacement versus time is a sine curve (Figure 4) $x = A\sin(\omega t)$, not a cosine curve.

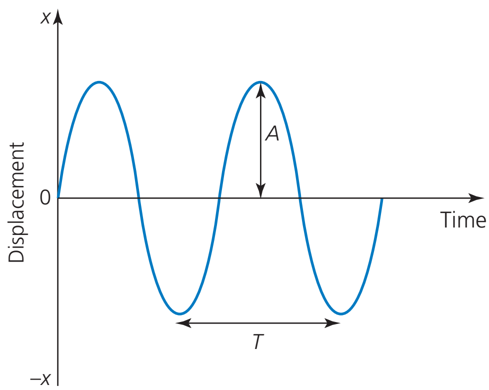

Figure 4 Displacement versus time (sine curve) for SHM with $x = 0$ when $t = 0$

QUESTIONS

1. Plot a displacement–time graph to show three cycles of an oscillation that has an amplitude of 6.0 mm and a frequency of 5000 Hz, assuming that displacement is a maximum when $t = 0$.
2. An object is oscillating with SHM. Assuming that its displacement $x = 0$ when $t = 0$, calculate the displacement after 0.24 ms if the amplitude is 6 mm and the frequency is 5000 Hz.
3. Sketch two cycles of a displacement versus time graph in which $x = 0$ when $t = 0$ for an oscillation of time period 0.5 s and amplitude 10 cm, showing appropriate scales on both axes.

Velocity and acceleration in SHM

Since velocity is defined as the rate of change of displacement, a graph of velocity versus time shows the variation with time of the gradient of the displacement–time graph (Figure 5). Similarly, an acceleration versus time graph can be obtained from the gradient of the velocity–time graph. A comparison of the graphs of displacement, velocity and acceleration versus time reveals the relationships between these three quantities during the oscillation (Figure 5).

- The velocity of the oscillating mass is zero when displacement is at a maximum, and at its greatest when displacement is zero.
- The acceleration is zero when the displacement is zero, and at its greatest when displacement is a maximum.

We can describe these differences in terms of **phase differences**. Phase difference can be expressed in terms of fractions of a cycle, or degrees or radians. To find the phase difference, for example between the displacement and the velocity, determine the time that elapses between each quantity being at a maximum. The phase difference in terms of a fraction of a cycle can then be found by dividing the time that elapses by the time period. Conversion to degrees or radians can be made by equating a full cycle to 360° or 2π radians. A comparison of the displacement and velocity graphs of Figure 5 shows a phase difference of a quarter of a cycle (90° or $\pi / 2$ rad) between the displacement and the velocity (*see section 5.2 in Chapter 5 of Year 1 Student Book*).

A comparison of the displacement and acceleration graphs reveals a phase difference of half of a cycle (180° or π rad) between these two quantities – they achieve their maximum values at the same instant but have opposite directions.

Simple harmonic motion is defined by the equation

$$a = -\omega^2 x$$

where a is the acceleration, x is the displacement of the oscillating object and ω is the angular frequency, which is a constant for the motion. The equation shows that the acceleration is directly proportional to displacement but in the opposite direction. The maximum value for the acceleration is given by

$$a_{max} = \omega^2 A$$

since the amplitude A is the maximum displacement.

The velocity of an object moving with SHM is given by

$$v = \pm 2\pi f \sqrt{A^2 - x^2}$$

with the maximum velocity $v_{max} = 2\pi f A = \omega A$, since the velocity is at its greatest at the equilibrium position which corresponds to $x = 0$.

A graph of acceleration a versus displacement x for an object oscillating with SHM (Figure 6) is a straight line through the origin with a gradient equal to $-\omega^2$. Since, for a constant mass, acceleration is directly proportional to the resultant force (from Newton's second law, $F = ma$), the restoring force is directly proportional to the displacement but in the opposite direction. Therefore a graph of the restoring force versus displacement would be a straight line through the origin with a gradient equal to $-m\omega^2$, where m is the mass of the oscillating object.

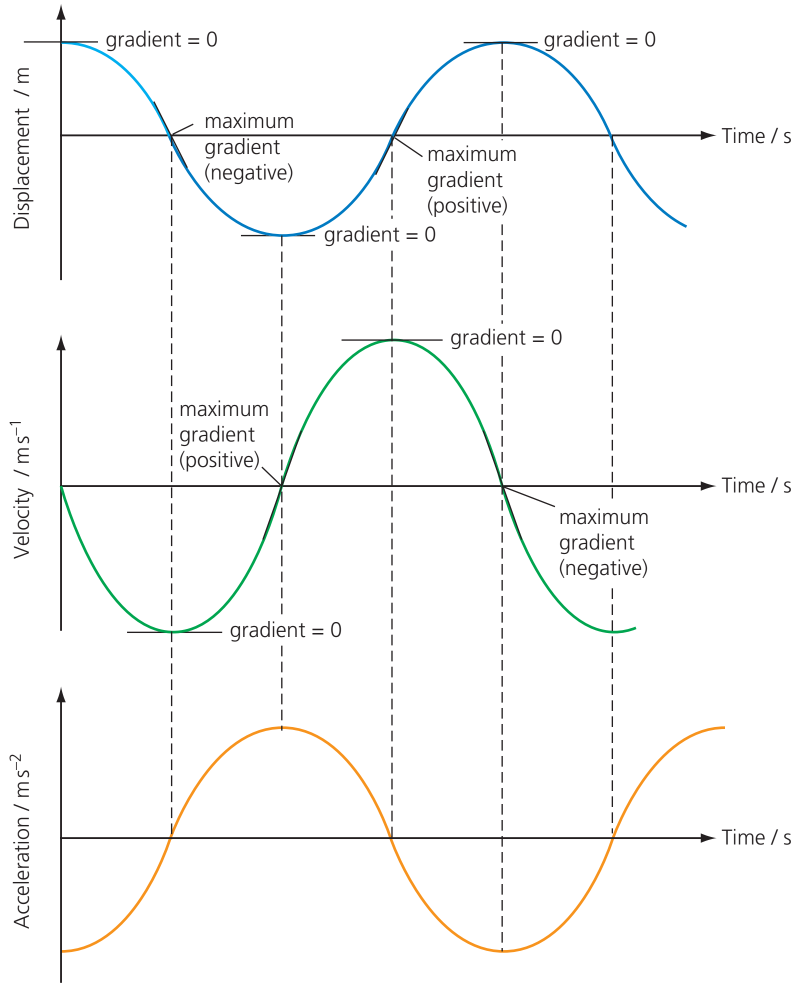

Figure 5 *Displacement, velocity and acceleration versus time graphs for a simple harmonic oscillator*

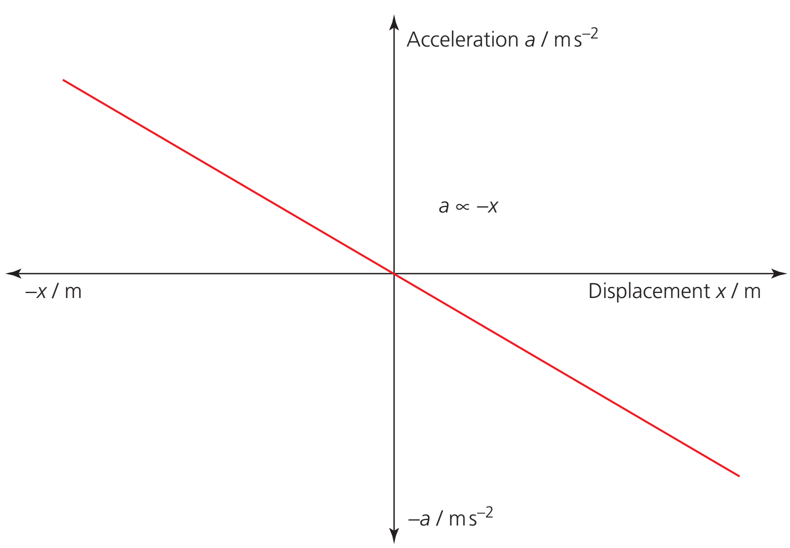

Figure 6 *Acceleration versus displacement for an object moving with SHM*

Worked example

A mass attached to a spring vibrates up and down, undergoing SHM with a time period of 1.6 s. The distance from the top to the bottom extreme positions is 8.0 cm. Determine the maximum acceleration and the speed of the mass as it passes through the equilibrium position.

Since, for SHM, acceleration $a = -\omega^2 x$, the maximum acceleration is

$$a_{max} = \omega^2 A = \left(\frac{2\pi}{T}\right)^2 A$$

The value of the amplitude A is half the distance from one extreme position to the other, so $A = 4.0$ cm. Therefore

$$a_{max} = \left(\frac{2\pi}{1.6}\right)^2 \times 0.040 = 0.62\,\mathrm{m\,s^{-2}}$$

Velocity $v = \pm 2\pi f\sqrt{A^2 - x^2}$, but at the equilibrium position, $x = 0$ and the velocity has its maximum value:

$$v_{max} = 2\pi f\sqrt{A^2} = \frac{2\pi}{T}A = \frac{2\pi}{1.6} \times 0.040 = 0.16\,\mathrm{m\,s^{-1}}$$

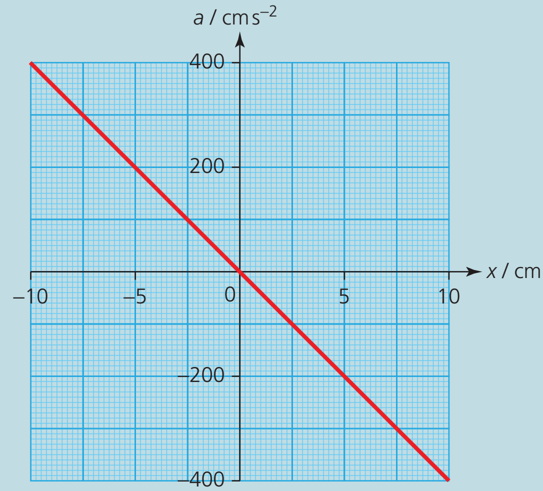

Figure 7

a. Determine the gradient of the line and write an equation showing the relationship between the object's acceleration and its displacement.

b. How would the magnitude of the gradient change if the oscillating system was changed so that the frequency was doubled?

QUESTIONS

4. The piston of a car engine moves with a motion that is approximately SHM. One cycle of motion takes 0.017 s and the piston moves through a total distance of 100 mm. Calculate the maximum acceleration of the piston and its velocity at a distance of 20 mm from its equilibrium position.

5. Determine the acceleration of an object oscillating with SHM at a frequency of 2.0 Hz when its displacement is 0.20 m.

6. A graph of acceleration versus displacement for an object moving with SHM is shown in Figure 7.

KEY IDEAS

- SHM is an oscillation in which the acceleration (a) is directly proportional to the displacement (x) but in the opposite direction:

$$a = -\omega^2 x$$

  where ω is the constant angular frequency of the motion.

- The displacement versus time graph of an oscillating object is a sine or cosine curve depending on whether the displacement is zero or at its maximum value at time $t = 0$.
- The velocity versus time graph can be determined from the gradient of the displacement versus time graph.
- The acceleration versus time graph can be determined from the gradient of the velocity versus time graph.

- The maximum velocity occurs as the object passes through the equilibrium position and is given by

$$v_{max} = \omega A$$

where A is the amplitude (maximum displacement) of the oscillation.

- The maximum acceleration occurs at the extremes of the oscillation and is given by

$$a_{max} = \omega^2 A$$

2.2 SIMPLE HARMONIC SYSTEMS

The mass–spring system

So why does a mass oscillating on the end of a spring move with SHM? First, here is a reminder of Hooke's law: 'The extension of a spring is directly proportional to the force applied provided the elastic limit has not been exceeded'. We can express this as

$$F = k\Delta l$$

where k is the spring constant and Δl is the extension (*see section 12.2 in Chapter 12 of Year 1 Student Book*).

Consider a spring that is initially suspended vertically but has no mass attached (Figure 8a). A mass m is then attached to the spring, extending it by an amount denoted by e (Figure 8b), with the mass then being at rest in its equilibrium position. Since the mass is at rest under the action of both the force of gravity, mg, and the tension in the spring ke, it follows that $ke = mg$.

Displacing the mass below its equilibrium position and then releasing it allows the mass to oscillate about its equilibrium position. The instant that the mass passes though the point that is a distance x below equilibrium is represented by Figure 8c. At this instant, the extension of the spring is given by $e + x$ and the tension in the spring is $k(e + x)$.

Ignoring the effects of air resistance on the oscillating mass, the resultant upward force F on the mass is equal to the tension minus the weight, which gives the equation

$$F = k(e + x) - mg$$

However, since $ke = mg$, the expression for the resultant force becomes $F = kx$.

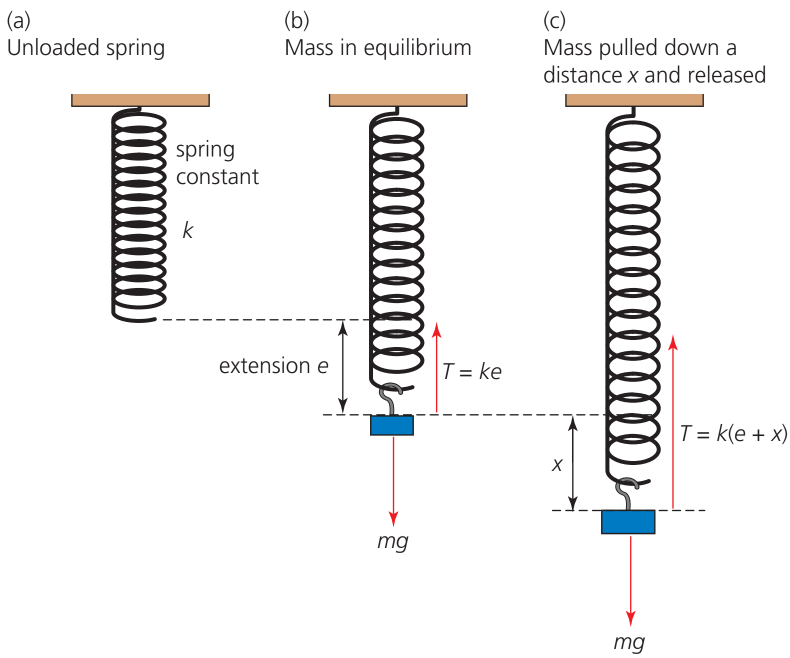

Figure 8 *The forces acting on a mass–spring system*

Since the resultant force is upwards when the displacement is downwards, the resultant force is written as

$$F = -kx$$

From Newton's second law ($F = ma$), the acceleration of the oscillating mass is therefore

$$a = -\frac{k}{m}x$$

Given that $\frac{k}{m}$ is constant, this shows that the acceleration of an oscillating mass is directly proportional to its displacement but in the opposite direction – which conforms to the definition of SHM.

Comparing the SHM defining equation $a = -\omega^2 x$ with $a = -\frac{k}{m}x$ shows that $\omega^2 = \frac{k}{m}$. Substituting $\omega = \frac{2\pi}{T}$ into the equation gives $\left(\frac{2\pi}{T}\right)^2 = \frac{k}{m}$, which rearranges to give the equation for the time period of a mass–spring system:

$$T = 2\pi\sqrt{\frac{m}{k}}$$

Worked example

A family car has a mass of 1000 kg when it is not loaded. This mass is supported equally by four identical springs. When the car is fully loaded, its mass goes up to 1250 kg and the springs compress by a further 2 cm. When the car goes over a bump in the road, it bounces on its springs. Find the time period of these oscillations.

The equation for the period of a mass–spring system is

$$T = 2\pi\sqrt{\frac{m}{k}}$$

We know the mass of the system, but we need to calculate the spring constant, k.

The extra weight of $250\,\text{kg} \times 9.81\,\text{N kg}^{-1} = 2450\,\text{N}$ will depress the four springs by 0.02 m.

Assuming each of the identical springs carries a quarter of the extra weight, 2450/4 = 613 N, we can use Hooke's law, $F = k\Delta l$, to find the spring constant k of one spring:

$$k = \frac{F}{\Delta l} = \frac{613}{0.02} = 3.06 \times 10^4\,\text{N m}^{-1}$$

Since the effective mass oscillating on each spring is 1250/4 = 313 kg, the time period T of the oscillation of the car is

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{313}{3.06 \times 10^4}} = 0.64\,\text{s}$$

QUESTIONS

7. A mass of 250 g is attached to a spring with a spring constant of $30\,\text{N m}^{-1}$. Determine the time period if the mass was displaced so that the mass–spring system oscillated.
8. Four identical springs, each with a spring constant of $40\,\text{N m}^{-1}$, are arranged so that two are joined in series (Figure 9a) and the other two in parallel (Figure 9b). A mass of 500 g is attached to both arrangements.

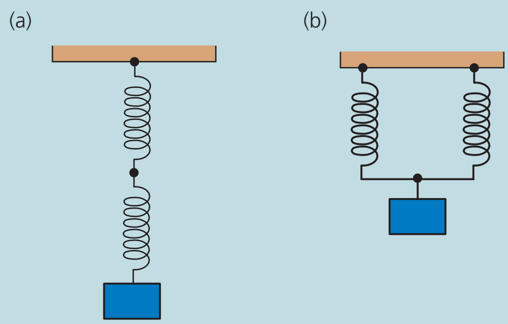

Figure 9 *Oscillating systems made up of springs (a) in series and (b) in parallel*

Determine the time period of oscillation of the arrangement of two springs

a. in series

b. in parallel.

[*Hint*: Combining two springs in series halves the spring constant. Combining two springs in parallel doubles the spring constant.]

9. Determine the phase difference in radians between two identical mass–spring systems, each with a time period of 1.2 s, if one system's oscillations are started 0.4 s before the other.
10. A mass of 200 g oscillates with a time period of 1.46 s when attached to a spring. Calculate the spring constant.
11. An oscillating mass–spring system has a time period T. If the mass is then doubled and the spring replaced so that the spring constant is halved, what is the new time period?

A $\sqrt{2}\,T$ **B** $2T$ **C** $\frac{T}{\sqrt{2}}$ **D** $\frac{T}{2}$

REQUIRED PRACTICAL: APPARATUS AND TECHNIQUES

Part 1: Investigation into simple harmonic motion using a mass–spring system

The aim of this practical is to test the equation $T = 2\pi\sqrt{\frac{m}{k}}$ and obtain a value for k, by carrying out measurements of time period for various masses attached to a helical spring. This practical gives you the opportunity to show that you can:

- use appropriate digital instruments to obtain a range of measurements (to include time)
- use methods to increase the accuracy of measurements, such as timing over multiple oscillations, or use of a fiduciary marker
- use a stopwatch for timing.

There are a number of ways of measuring the time period, including using digital or analogue stopwatches, light gates, or a motion sensor with a data logger. The method described here involves timing oscillations using a digital stopwatch and a fiducial marker.

Apparatus

A digital stopwatch with a precision of ±0.01 s is used, and a fiducial marker to help with counting oscillations.

A helical spring is clamped securely and supported by a stand (Figure P1). A fiducial marker in the form of an optical pin is inserted into a cork and supported in a clamp and stand. The purpose of this is to indicate the mean (equilibrium) position of the oscillation. A range of standard masses are used, accurate to within ±2 g and suitable for the choice of spring.

Techniques

Once a mass has been attached to the spring, the fiducial marker is aligned with the equilibrium position of the mass–spring system to enable accurate counting of cycles of oscillation. One cycle of oscillation corresponds to the mass passing the equilibrium position, moving to one extreme, passing the equilibrium position again, moving to the other extreme, then back to the equilibrium position. Although a full cycle also corresponds to the mass moving from one extreme to the other extreme and then back again, it is better to count oscillations with

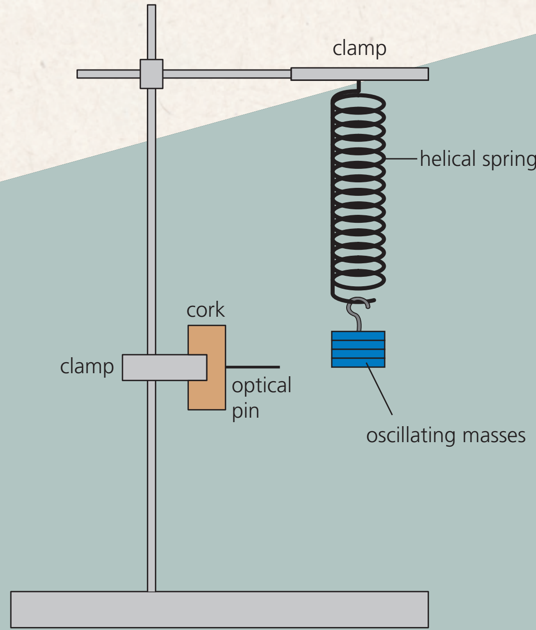

Figure P1 *Set-up for the oscillating mass–spring experiment*

respect to the equilibrium position because this does not change, whereas the extreme positions change as the oscillation loses energy.

The mass should be raised and then released so that it starts oscillating. Once the oscillation is established, the digital stopwatch should be started as the mass passes the fiducial marker. Twenty cycles of oscillation are then counted and the stopwatch should be stopped as the mass passes the fiducial marker completing its 20th cycle of oscillation. Two repeat measurements are taken and the average of the three measurements is calculated. The time period is determined by dividing the average time for 20 cycles of oscillations by 20. The process is repeated using a range of masses.

Analysis

The time period equation for the mass–spring system $T = 2\pi\sqrt{\frac{m}{k}}$ can rewritten as

$$T^2 = \frac{4\pi^2}{k}m$$

which can be compared with the equation of a straight line $y = mx + c$ (where m here is the constant gradient, not the variable mass). If a graph of T^2 as the y variable and mass m as the x variable is plotted with an origin (0, 0), then the theory predicts that the graph should be a straight line through the origin. The gradient of the line is equal to $\frac{4\pi^2}{k}$, which enables a value for the spring constant k to be determined.

The uncertainty in the time for 20 oscillations for a particular mass can be found from half of the range of the repeat measurements. The presence of reaction time error suggests that the uncertainty in the time for 20 oscillations cannot be less than typically ± 0.2 s. The uncertainty in time period T is found by dividing the uncertainty in 20 oscillations by 20. The percentage uncertainty in T can then be calculated. The percentage uncertainty in T^2 is twice the percentage uncertainty in T (*see section 1.4 in Chapter 1 of Year 1 Student Book*).

The uncertainty in the value of standard masses of 50 or 100 g is typically about ± 2 g. As more masses are added to the hanger, this uncertainty accumulates, but the percentage uncertainty for the total mass is unchanged. The uncertainties in T^2 and m enable error bars to added to the points on the T^2 versus m graph, so that the best line, and the steepest gradient and the shallowest gradient lines that fit within the error bars can be drawn. The percentage uncertainty in the value of k is equal to the percentage uncertainty in the gradient which can be calculated from

$$\text{percentage uncertainty} = \frac{\text{best gradient} - \text{worst gradient}}{\text{best gradient}} \times 100\%$$

where the worst gradient is whichever of the steepest or shallowest gradient values differs from the best gradient by the largest amount.

QUESTIONS

P1 Discuss whether reaction time error when measuring the time for 20 oscillations is random or systematic.

P2 A student measures time periods for a mass–spring system for various masses using the method described. One of her data sets is shown in Table P1. The repeat times for 20 cycles and the average time for 20 cycles have been recorded to the same precision as the digital stopwatch that was used to measure them.

a. i. Determine the uncertainty in the average time for 20 cycles from the range of the three $20T$ measurements.

ii. How does your value from part **i** compare with a typical reaction time error of ± 0.2 s?

b. Suggest an appropriate value for the uncertainty in time period T based on your answers to parts **a i** and **ii**, and record T to the appropriate number of significant figures.

c. i. The student decides to plot her full set of data in the form of a graph of T^2 versus mass and then obtain a value for the spring constant k. What would be the uncertainty in the plotted value of T^2 for the data set shown in Table P1?

ii. What source(s) of uncertainty would have the greatest effect on the maximum and minimum gradients, and hence the uncertainty in k?

iii. Would this still be the case if only 10 oscillations had been counted?

P3 Suggest whether or not a method using a motion sensor and data logger would give a more reliable test of the period–mass relationship and a more accurate value of k.

Mass attached, including hanger, m / kg	Time for 20 cycles, $20T_1$ / s	Repeat time for 20 cycles, $20T_2$ / s	Repeat time for 20 cycles, $20T_3$ / s	Average time for 20 cycles, $20T$ / s	Time period, T / s
0.150	9.86	9.64	9.72	9.74	

Table P1

The simple pendulum

The simple pendulum, consisting of a length of thread with a metal ball called a bob attached, usually made of lead or brass, has been an object of scientific study since the 17th century. Galileo is generally considered to be the first to observe that the time period of a swinging pendulum remained constant even as the oscillations died away (Figure 10). Another 17th century Italian astronomer, Giovanni Riccioli, devised a technique based on the oscillations of a pendulum to obtain the first accurate value for the acceleration due to gravity.

Figure 10 *A model of Galileo's proposed design for a pendulum clock. Galileo realised that a swinging pendulum could be used for time-keeping. This was a major step in the history of the development of clocks.*

A free-body force diagram of an oscillating simple pendulum (Figure 11) shows that the forces acting on the pendulum bob are the tension T in the string and the weight W of the bob, assuming that air resistance is negligible.

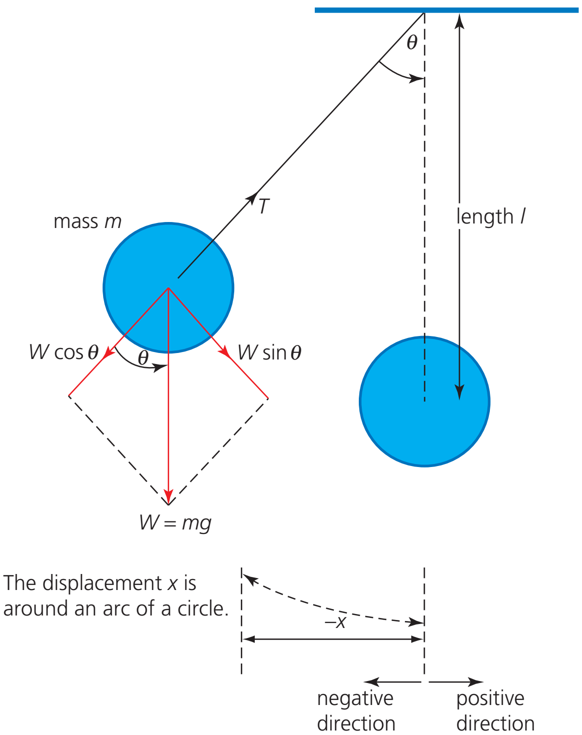

Figure 11 *The forces acting on the bob of a simple pendulum*

The weight of the bob is resolved into two components (*see section 10.3 in Chapter 10 of Year 1 Student Book*), one parallel to the tension and the other at 90° to the tension. Since the displacement x of the bob is actually along the arc of a circle with radius equal to the length l of the pendulum, the angle θ in radians is equal to $\frac{x}{l}$. The restoring force F is provided by the component of the bob's weight that acts at 90° to the tension, and therefore

$$F = W \sin\theta = mg \sin\theta$$

The small-angle approximation states that, for small angles (<10°), $\sin\theta \approx \theta$ in radians. If this is applied to the above equation, the expression for the restoring force becomes

$$F = mg\theta$$

Substituting $\theta = \frac{x}{l}$ gives $F = mg\frac{x}{l}$. But since the restoring force is always in the opposite direction to the displacement, the expression for the restoring force becomes

$$F = -mg\frac{x}{l}$$

From Newton's second law, the acceleration of the oscillating pendulum is therefore

$$a = -\frac{g}{l}x$$

This equation shows that the acceleration of an oscillating simple pendulum of constant length is directly proportional to and in the opposite direction to its displacement, *providing only small-amplitude oscillations are considered*. In other words, the simple pendulum oscillates with SHM.

Comparing $a = -\frac{g}{l}x$ with the defining equation for SHM, $a = -\omega^2 x$, shows that $\omega^2 = \frac{g}{l}$. Substituting $\omega = \frac{2\pi}{T}$ into the equation gives $\left(\frac{2\pi}{T}\right)^2 = \frac{g}{l}$, which rearranges to give the equation for the time period T of a simple pendulum:

$$T = 2\pi\sqrt{\frac{l}{g}}$$

The theory shows that the time period of a simple pendulum depends only on its length l provided angles of swing are less than about 10°. The mass of the pendulum bob has no effect on the time period provided the bob's mass is much greater than the mass of the string it is attached to. Although air resistance will remove energy from the oscillation, causing the amplitude to decrease, the time period of the oscillation is unaffected.

QUESTIONS

12. Calculate the length of a simple pendulum that would have a time period of 1.0 s.

13. A simple pendulum has a time period of T. What would be its new time period if its length is doubled?

A $2T$ **B** $\sqrt{2}\,T$ **C** $\frac{T}{\sqrt{2}}$ **D** $\frac{T}{2}$

KEY IDEAS

- The equation for time period T of a mass–spring system is

$$T = 2\pi\sqrt{\frac{m}{k}}$$

- The equation for the time period T of a simple pendulum for small-amplitude oscillations is

$$T = 2\pi\sqrt{\frac{l}{g}}$$